Option pricing with stable-like processes in stochastic volatility models and its optimization.

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Declaration

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Abstract

In this thesis we are concerned with the optimal control of jump type Stochastic Differential equations(SDEs), which we utilize to model the incomplete financial market. It is demonstrated how Lévy type processes associated to general generators with variable coefficients can be linked to Lévy processses with jumps in an abstract setting. Thus providing a useful way to deal with optimization problems where the financial market is being driven by a Lévy type process. An application to two portfolio optimization problems will be made, initially set out in the abstract setting. Then with a special interest in modelling with stable-like processes we construct the coefficient of the jump term in the associated jump-type SDE, associated to a polar decomposed Lévy measure, we are able to solve these optimization problems concretely.

Contents

Chapter 1

Introduction

In 1973 Black and Scholes presented the classical model for the asset price which evolved according to a geometric Brownian motion. They produced the Black-Scholes equation, which provided a quantitative instrument for calculating the prices of options in which the determining variable is the volatility of the underlying asset.

1.1 The Black-Scholes framework.

To begin let us oultine the assumptions set upon this model.

1. The asset price follows a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1.1}$$

   where μ represents the drift and the average rate of return of the asset at time t, σ represents the volatility of the asset (i.e models the random change in the asset price).

2. The risk-free interest rate r and the asset volatility σ are known functions of time over the life of the option.

3. There are no transaction costs associated with hedging a portfolio.
4. The underlying asset pays no dividends during the life of the option. This assumption can be dropped if the dividends are known beforehand. They can be paid either at discrete time intervals or continuously over the life of the option.
5. There are no arbitrage opportunities. The absence of arbitrage opportunities means that all risk-free portfolios must earn the same return.
6. Trading of the underlying asset can take place continuously.
7. Short selling is permitted and the assets are divisible. We assume that we can buy and sell any number (not necessarily an integer) of the underlying asset, and that we may sell assets that we do not own.

Now from Eqn (1.1) it is possible to derive the celebrated Black-Scholes differential equation for the price of options. So suppose that we have an option whose value is defined by $V(S,t)$. Now, this can also be the value of a whole portfolio of different options. Using Itô's lemma, we can write

$$dV = (\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t})dt + \sigma S \frac{\partial V}{\partial S} dW. \tag{1.2}$$

This gives the stochastic process followed by V. We require V to have at least one time derivative and two spatial derivatives. The next step, we construct a portfolio consisting of one option and a number $-\delta$ of the underlying asset. This number is unspecified. Note that δ is the rate of change of the value of the option or portfolio of options with respect to S. Basically, its a measure of the correlation between movements of the option or other derivative products and those of the underlying asset. Let the value of the portfolio be defined

$$\Pi_t = V - \delta S. \tag{1.3}$$

The jump in the value of the portfolio in one time-step is

$$d\Pi_t = dV - \delta dS. \tag{1.4}$$

Here δ is held fixed during the time-step. Substituting (1.1) and (1.2) into (1.4), we get

$$d\Pi_t = (\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu\delta S)dt + \sigma S(\frac{\partial V}{\partial S} - \delta)dW.$$

If we choose $\delta = \frac{\partial V}{\partial S}$, then the second term on the right hand side in the above formula is zero. Therefore the above can be reduced to

$$d\Pi = (\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t})dt. \tag{1.5}$$

Notice that the above formula does not contain the random components. Now we can see that choosing

$$\delta = \frac{\partial V}{\partial S}$$

provides the advantage that we can reduce the stochastic expression into a deterministic expression.

We now appeal to the concepts of arbitrage and supply and demand, with the assumption of no transaction costs. The return on an amount Π invested in risk-less assets would see a growth of $r\Pi dt$ in a time dt. If the right hand side of (1.5) were greater than this amount, $r\Pi dt$, an arbitrager could make a guaranteed risk-less profit by borrowing an amount Π to invest in the portfolio. The return for this risk-free strategy would be greater than the cost of borrowing. Conversely, if the right hand side of (1.5) were less than $r\Pi dt$, then the arbitrager would short the portfolio and invest Π in the bank. Either way the arbitrager would make a risk-less, no cost, instantaneous profit. The existence of such arbitragers with the ability to trade at low cost ensures that the return on the portfolio and on the risk-less account are more or less equal. Thus, we have

$$r\Pi dt = (\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t})dt. \tag{1.6}$$

Now replace Π by $V - \delta S$, and replace δ by $\frac{\partial V}{\partial S}$ and then divide both sides by dt, we obtain the celebrated Black-Scholes differential equation.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{1.7}$$

It should be understood that any derivative security whose price depends on the current value of S and t, which satisfy the assumptions previously stated, which is paid for up front, must satisfy the Black-Scholes equation (1.7). From this idea the Black-Scholes formula was derived. Suppose a European call option $c(S,t)$ satisfies (1.7) and is subject to boundary conditions

$$c(0,t) = 0 \quad c(S,t) \sim S \quad as \quad S \to \infty,$$

and a terminal condition

$$c(S,T) = max(S(T) - E, 0).$$

The solution can then be expressed as follows

$$c(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\ln(\frac{S}{E}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

and

$$d_2 = \frac{\ln(\frac{S}{E}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

For a put option, the solution is

$$p(S,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1).$$

In doing all this it brought about the use of Itô stochastic calculus and the Markov property of diffusions. Essentially the work of Black et al. brought order to rather chaotic situation, where the previous pricing of options had been done solely by intuition about ill-defined market forces.

1.2 Stochastic volatility

Moving on from the Black-Scholes framework, it is widely accepted that the assumption of a constant volatility is not realistic for modelling asset returns. A major obstacle in option pricing is the problem of calibrating the parameters to the current skew (i.e the theoretical option prices should agree with the market prices for different strikes.) The Black-Scholes framework of a log normal distribution fails to explain the existence of fat tails as well as the asymmetry observed. Ultimately failing to explain long observed anamolies such as the volatility smile, which indicate that the volatility does tend to vary over time. By assuming that the volatility of the asset price is a stochastic process rather than a constant, it becomes possible to more accurately model options.

Starting from (1.1), we can get a stochastic volatility model by replacing the constant σ by a function v_t, that models the variance of the price of the asset S_t. This variance function is also modeled as a Brownian motion, and the form of v_t depends on the particular stochastic volatility model under study. Let us consider the basic model where

$$dS_t = \mu S_t dt + v_t S_t dW_t,$$

and

$$dv_t = \alpha_{S,t}dt + \beta_{S,t}dB_t,$$

where $\alpha_{S,t}$ and $\beta_{S,t}$ are functions of v.

Popular stochastic volatility models include GARCH, Jump diffusions, Heston and Variance-gamma models.

The Heston model.

This model considers the randomness of the variance process as the square root of the variance. Therefore, the stochastic differential equation for the variance takes the form

$$dv_t = (w - \theta v_t)dt + \epsilon\sqrt{v_t}dB_t,$$

where w is the mean long-term volatility, θ is the rate at which the volatility reverts toward its long-term mean, ϵ is the volatility process, and dB_t is, like dW_t, a gaussian with zero mean and unit standard deviation. However, dW_t and dB_t are correlated with the constant correlation coefficient ρ.

In other words, this Heston model assumes that volatility is a random process that

1. exhibits a tendency to revert towards a long-term mean volatility w at a rate θ,
2. exhibits its own constant volaitlity ϵ,
3. and whose source of randomness is correlated(with correlation ρ) with the randomness of the price process.

GARCH model

The Generalized Auto-Regression Conditional heteroskedacity (GARCH) model is another popular model for estimating stochastic volatility. It assumes that the randomness of the variance process varies with the variance, as opposed to the square root of the variance in the Heston model. The Standard GARCH model has the following form for the variance differential

$$dv_t = (w - \theta v_t)dt + \epsilon v_t dB_t.$$

Essentially, the idea behind stochastic volatility modelling is once a particular model is chosen, it will be calibrated against existing market data.

However, one of the main difficulties while working with a stochastic volatility model is that the actual instantaneous volatility is not observable in the market and therefore needs to be modelled as a hidden state. This means that in order to calibrate a model to the financial market, one would require to use usually a non-linear and/or a non-Gaussian filter. This calibration would then provide an estimate of the statistical (or real world) distribution of the financial market.

As a consequence there has been interest in modelling financial markets which are under the influence of non-Gaussian stochastic volatilities. This is due to the Gaussian distribution and linear dynamic assumptions being unsatisfactory in modelling the heavy tails, skewness (i.e asymmetry) and time changing volatility. Therefore the attention has been turned to non-linear and non-Gaussian models, however they are generally quite difficult to handle and present challenging problems in application.

A particularly well known non-Gaussian stochastic volatility model is that presented by O.Barndorff-Nielsen and N.Shephard, namely their Lévy driven stochastic volatility model in [4]. Another example is the Lévy driven stochastic volatility considered by Carr, Geman, Madan and Yor in [14] where they consider a time changed Lévy process. These such models are based around infinately active pure jump Lévy processes.

1.3 Motivation

With the Black-Scholes framework being the first essential tool for financial modelling it wasn't until later that decade when the theory of stochastic integration for semimartingales was developed due in large part to P.A Meyer of Strasbourg and his collaborators. These advances were then combined to the work of Black et al. to further advance the theory of Harrison and Kreps in 1979 and that of Harrison and Pliska in 1981 in articles published in 1979 and 1980. In particular they established a connection between complete markets and martingale representation.

Much has happened in the last twenty years, and the subject has attracted the interest of many mathematicians, the interweaving of finance and stochastic integration continues today and will continue into the future. As is clear the Black-Scholes model driven by a geometric Brownian motion is a little inconsistent with the calculating of options and asset prices, this is due to it being unable to manage sudden changes in variation which tend to be caused by unforeseen events or occurences such as earthquakes, wars, decisions of the Federal Reserve etc. This as a result caused mathematicians, economists to turn their attention to models which are able to manage such events. It turned out that models which consisted of a diffusion part and a pure jump process would suffice, thus making Lévy processes with jumps the ideal candidate. Lévy processes in finance are a useful tool due to its distributional flexibility, they are able accomodate heavy tailed distributions, plus generalized Lévy processes can accomodate jumps, while at the same time allowing enough analytical tractability.

It seems that models in finance using jump Lévy processes can be seperated into two groups.

1. Jump diffusion models - where the jumps are specified to model rare events and the jump component is of finite activity.

2. Models based on infinitely active jump Lévy processes. Namely those mentioned above in section 1.2, the Lévy driven stochastic volatility models.

More recently, there has also been interest in using different types of Lévy processes to model the financial markets. This being said we have chosen as an area of interest to consider the market models under the influence of more general Lévy type processes, where the stable-like process is a special case.

Along with this tendency, there has been considerable research interest in the optimal control of the financial markets. Optimal control theory is a mathematical field which is concerned with control policies that can be deduced using optimization techniques. It deals with the problem of finding a control law for a given system such that certain optimality conditions are acheived. These conditions come in the form of equations which describe the evolution of the parameters defining the system. A characteristic example of this is the Hamilton-Jacobi-Bellman equation.

The particular advantage of this approach consists in the fact that it allows the researchers to view the patterns of the system in consideration.

The Optimization technique refers to a study of problems in which one seeks to maximize/minimize a real function by systematically choosing the values of real or integer variables from an allowed set. It is this particular technique which is increasingly being employed in financial mathematics, for example in portfolio selection problems, when considering the pricing of stock options, risk management and many more. I envisage this technique will become more and more apparent not only in financial mathematics but in other areas of investigation like biology, physics, engineering etc. There is extremely interesting text [40] which considers optimal control, impulse control and optimal stopping applied to numerous different areas of mathematics. However this text has its focus more on the applied aspect of the theory. Thus it is natural for us to consider the optimal control of the financial markets driven by a Lévy type process.

1.3.1 Lévy process versus Lévy type process.

As is fairly clear our interest is in the modelling of the financial market driven by Lévy type processes, and to consider the optimal control of jump type Stochastic Differential equations(SDEs). Before we proceed it is necessary that we outline the important properties of Lévy type processes. With them being closely related to Lévy processes we will start by defining a Lévy process.

Definition 1.3.1 *An adapted process $\{X_t\}_{t\geq 0}$ with $X_0 = 0$ a.s. is called a Lévy process if X_t is continuous in probability and has stationary and independent increments.*

The next important result we will consider is the beautiful formula, first established by Paul Lévy and A.Ya.Khinchine in the 1930s, namely the Lévy-Khinchine formula. This formula gives a characterisation of infinitely divisible random variables through their characteristic functions.

Let us take the characteristic exponent(or Lévy symbol) $\phi : \mathbb{R}^d \to \mathcal{C}$ given by

$$E[e^{iuX_t}] = e^{t\phi(u)},$$

where the characteristic function is defined $\mu_t(u) = e^{t\phi(u)}$. It is well known that ϕ has the following Lévy-Khinchine representation

$$\phi(u) = ibu - \frac{1}{2}uQu^T + \int_{\mathbb{R}^d \setminus \{0\}} \{e^{iuz} - 1 - iuz.1_{\{0<|z|<1\}}(z)\}\nu(dz),$$

where $b \in \mathbb{R}^d, u \in \mathbb{R}^d$, Q is a non-negative definite symetric matrix and a Lévy measure ν on $\mathbb{R}^d \setminus \{0\}$ satisfies

$$\int_{\mathbb{R}^d \setminus \{0\}} \frac{|z|^2}{1+|z|^2}\nu(dz) < \infty.$$

Thus making (b, Q, ν) the characteristics which describe a Lévy process.

From the Lévy-Khinchine formula it is possible to obtain an expression for the infinitesimal generator L, for each $f \in C^2(\mathbb{R}^d), x \in \mathbb{R}^d$

$$\begin{aligned} Lf(x) &= b^i\partial_i f(x) + \frac{1}{2}q^{ij}\partial_i\partial_j f(x) \\ &+ \int_{\mathbb{R}^d \setminus \{0\}} \{f(x+z) - f(x) - z^i\partial_i f(x).1_{\{0<|z|<1\}}(z)\}\nu(dz). \end{aligned}$$

It is well known that the process generated from this generator is called a Lévy process.

The final key result from the theory of Lévy processes is the celebrated Lévy-Itô decomposition. It is well known that the Lévy characteristics (b, Q, ν) imply a path decomposition of the trajectories $t \to X_t(\omega)$. These are called the sample paths, and the decomposition splits them into continuous and jump parts respectively. It is important to get a probabilistic interpretation of the Lévy-Khinchine formula, and this is what this decomposition does. Fundamentally, it describes the way that the measure ν determines the structure of the jumps in the process. Specifically it states the Lévy process X_t has a decompostion

$$X_t = bt + \sigma W_t + \int_{0<|z|<1} x\tilde{N}(t,dz) + \int_{|z|\geq 1} xN(t,dz),$$

for $b \in \mathbb{R}^d$,$a \in \mathbb{R}^{d\times m}$, W_t is an m-dimensional Brownian motion and

$$\tilde{N}(dt,dz) := N(dt,dz) - \nu(dz)dt$$

is the compensated $\{\mathcal{F}_t\}_{t\geq 0}$- martingale measure of N, the poisson random measure of X_t.

We now move to considering the more general jump type Markov processes, i.e a Lévy type process. If we define Y_t to be a Lévy type process. The generator associated to a Lévy type process is similar to the generator previously discussed however the coefficients contained (b, q, ν) are now variable and are dependent on the starting point $y \in \mathbb{R}^d$. So let us define the Lévy type generator L, for each $f \in C^2(\mathbb{R}^d), y \in \mathbb{R}^d$

$$\begin{aligned} Lf(y) &= b^i(y)\partial_i f(y) + \frac{1}{2}q^{ij}(y)\partial_i\partial_j f(y) \\ &+ \int_{\mathbb{R}^d\setminus\{0\}} \{f(y+z) - f(y) - z^i\partial_i f(y).1_{\{0<|z|<1\}}(z)\}\nu(y,dz), \end{aligned}$$

as a result of having variable coefficients, we lose the translation invariance property that a Lévy process posses.

A Lévy type process differs from a Lévy process in the way that it is not spatially homogeneous. Therefore, the characteristic function of Y_t will now depend on the starting point y,

$$\mu_t(y,u) = E(e^{iuY_t}) = e^{t\phi(y,u)}.$$

As a result the Lévy-Khinchine representation for the characteristic exponent $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathcal{C}$ becomes

$$\phi(y,u) = ib(y)u - \frac{1}{2}uQ(y)u^T + \int_{\mathbb{R}^d \setminus \{0\}} \{e^{iuz} - 1 - iuz.1_{0<|z|<1}(z)\}\nu(y,dz).$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$, $Q : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ and the Lévy kernel satisfies

$$\int_{\mathbb{R}^d \setminus \{0\}} \frac{|z|^2}{1+|z|^2}\nu(y,dz) < \infty,$$

for each $y \in \mathbb{R}^d$.

From this brief excursion into the properties of Lévy processes and Lévy type processes it can be seen that the main difference is in the dependence of the state, where each characteristic involved now evolves according to the spatial variable. The properties that have been outlined here are those which are specific to this research, there is a lot more that could be said about Lévy and Lévy type processes but this will then go beyond the scope of the research, for more depth one could refer to [28].

Finally, as previously stated we have a specific interest in modelling with stable-like processes. So we will take this opportunity to outline the preliminaries of such a process. These conditions and characteristics will be needed when we consider the financial market model driven by a stable-like process.

1.3.2 Stable-like processes

The aim here is to introduce the theories and properties of stable-like proceses, such that we can fully understand what will be driving the financial market models to be considered later on. We have already come across some of the properties that will be outlined, however with a special interest in stable-like processes we will require more conditions and a more concrete understanding.

For this we will follow the line of [5], where the author considers the existence and uniqueness of a solution to the martingale problem associated with an infinitesimal generator of a pure jump Markov process.

We start with a stable-like process Y_t which is a pure jump processes with no gaussian component and zero drift(i.e determined by only the jump part of a jump-type SDE) with a Lévy measure which has a stable-like representation as follows

$$\nu(y, dz) = \frac{dz}{|z|^{d+\alpha(y)}}, \quad \alpha : \mathbb{R}^d \to (0, 2). \tag{1.8}$$

Now the associated generator of such a process is defined. For $f \in C^2(\mathbb{R}^d)$,

$$Lf(y) = \int_{\mathbb{R}^d \setminus \{0\}} \{f(y+z) - f(y) - z^i \partial_i f(y).1_{\{0<|z|<1\}}(z)\} \nu(y, dz).$$

A characterisation of a stable-like process Y_t is given by the Lévy Khinchine formula which is indeed a special case since we are only considering the jump part with the Lévy measure as defined by (1.8). This measure models the behaviour of the jump component and determines the frequency and magnitude of the jumps.

$$\begin{aligned} \Phi(y, u) &= \int_{0<|z|<1} (e^{iuz} - 1 - iuz) \nu(y, dz) \\ &+ \int_{|z| \geq 1} (e^{iuz} - 1) \nu(y, dz) \end{aligned} \tag{1.9}$$

where the Lévy measure satisfies

$$\int_{\mathbb{R}^d \setminus \{0\}} \frac{|z|^2}{1 + |z|^2} \nu(y, dz) < \infty.$$

Similarly to section 1.3.1 $\Phi(y, u)$ is known to be the characteristic exponent of Y_t and notably the characterstic function of a stable-like process is therefore defined

$$\mathbb{E}^x[e^{iuY_t}] = e^{t\Phi(y,u)} = e^{t|u|^{\alpha(y)}},$$

where $\alpha(y)$ is a function dependent on the initial starting point, $\alpha : \mathbb{R}^d \to (0, 2)$.

With the aim being to outline the properties of a stable-like process, the importance lies with the index being dependant on a spatial parameter, thus we state some conditions on this index such that the existence and uniqueness of a solution to the martingale problem hold. The conditions become significantly simpler when considering a stable-like process as opposed to those in [5] where the author considers a pure jump Markov process.

With the generator L, a probability measure P solves the martingale problem starting at $y_0 \in \mathbb{R}^d$ if $P(Y_0 = y_0) = 1$ and

$$f(Y_t) - f(Y_0) - \int_0^t Lf(Y_s)ds$$

is a martingale whenever $f \in C^2(\mathbb{R}^d)$. Note that the solution to this is not a process, it's a probability!

Now the first condition we set upon the index $\alpha(y)$ is this regularity condition;

$$0 < \inf_{y \in \mathbb{R}^d} \alpha(y) \leq \sup_{y \in \mathbb{R}^d} \alpha(y) < 2, \tag{1.10}$$

this inequality is important since it guarantees the process is not degenerate implying no singularities and no gaussianity(i.e when $\alpha(y) = 2$). However, under considerably stronger smoothness assumptions the index $\alpha(y)$ of a stable-like process is required to be Lipschitz continuous, more detail will be given regarding this in Chapter 4.

Now for the existence of a solution to the martingale problem we need the continuity of $\alpha(y)$ and for the uniqueness we need the Dini continuity of $\alpha(y)$. We will highlight the existence and uniqueness of a solution to the martingale problem by means of two theorems

Theorem 1.3.2 *By [5] suppose*

(1) $\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d\setminus\{0\}}\frac{|z|^2}{1+|z|^2}\nu(y,dz)<\infty$, *and*

(2) for each $f\in\mathbb{C}_b^2(\mathbb{R}^d)$, $Lf(y)$ *is uniformly continuous in* y.

Then for every $y_0\in\mathbb{R}^d$ *there exists a solution to the martingale problem for* A *starting at* y_0.

The proof of this is similar to that outlined in [5].

For the uniqueness we need $\alpha(y)$ to be Dini continuous, this means that there exists a β such that

$$|\alpha(y)-\alpha(y')|\leq\beta(|y-y'|)\quad\forall y,y'$$

and

$$\int_0^{\epsilon}\frac{\beta(y)}{y}dy<\infty\quad\forall\epsilon>0.$$

Theorem 1.3.3 *By [5] suppose* $\alpha(y)$ *is Dini continuous adhering to (1.10). If* L *is the generator with Lévy measure defined by (1.8), then there exists a unique solution to the martingale problem for* L *starting at* $y_0\in\mathbb{R}^d$.

Other papers that deal with properties of processes associated to generators with variable coefficients without assuming a great deal of smoothness include [6, 38, 48, 35, 32, 49].

Now that we have outlined the necessary preliminaries we can move to discussing what lies ahead in each chapter of this thesis.

There are three chapters to be considered. The feature of chapter 2 is to introduce the notions and important aspects of stochastic control theory applied to jump type Markov processes determined by jump-type SDEs. The sufficient maximum principle will be given and the connection between this and Dynamic programming will be made. We will consider two optimization problems in the financial market where the sufficient maximum principle and the Dynamic programming techniques will be called on to solve them. Along with this we will take a brief excursion into calculating the prices and values of European style options for different situations within the Black-Scholes framework.

In chapter 3 we consider a two dimensional market model, where we take a wealth process which is based on the price of an asset at time t, where this is determined by the associated jump type SDE. The aim is to control the wealth and the average past consumption process with some decay rate by using a specific parameter process.

This type of problem is widely referred to as Merton's optimal investment problem, see [37], sometimes also referred to as the investment/consumption problem. Many researches have made extensive inroads to the study of stochastic models, to name a few, Pham [41], Benth et al. [7] - [9] and Framstad et al.[19].

The set-up that we will consider will consist of a wealth process and a cumulative consumption process where the value function depends on the consumption rate. The main results of this chapter are presented by means of an existence theorem and a uniqueness theorem, where the proof of the existence will be outlined.

The feature of chapter 4 is to study the optimal control of jump-type SDE's driven by stable-like processes, and to give its application to two financial optimization problems. We will start with a similar market model as used throughout, where we treated jumps abstractly over the σ-finite measure space $(U, \mathcal{B}(U), \lambda)$. We will construct the coefficient of the jump term by means of a polar decomposition for the concrete situation when dealing with a stable-like Lévy measure. Then bridge back into the σ-finite measure space $(\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \nu)$ with the aid of (4.2), consequently allowing us to solve the two financial optimization problems from the previous chapters. However, before they were simply considered abstractly, with the construction of the coefficient when dealing with the Lévy measure of a stable-like process we can solve them in this concrete setting.

The principal aim of this research is to investigate the optimal control of jump-type SDEs which we apptly use to model how the price of an asset changes as time changes. We also aim to show under what conditions can modelling with a Lévy type process be linked to a Lévy processes with jumps.

Chapter 2

The Stochastic control of SDEs and applications to financial optimization.

2.1 Preliminaries on Lévy generators and jump type SDEs

As is well known, a fairly large class of Markov processes on $\mathbb{R}^d$ are governed by Lévy generators. A Lévy generator L is a (time and space dependent) second order elliptic integro-differential operator (with variable coefficients) involving a diffusion matrix, a drift vector and a Lévy kernel. For certain "nice" functions $f : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$, for instance, $f \in C_b^{1,2}([0, \infty) \times \mathbb{R}^d)$, the operator L has the following representation

$$Lf(t,x) := \frac{1}{2}\sum_{i,j=1}^{d} a^{i,j}(t,x)\partial_i\partial_j f(t,x) + \sum_{i=1}^{d} b^i(t,x)\partial_i f(t,x) \tag{2.1}$$
$$+ \int_{\mathbb{R}^d\setminus\{0\}} \{f(t,x+z) - f(t,x) - \frac{z\cdot\nabla f(t,x)}{1+|z|^2}\}\nu(t,x,dz),$$

$(t,x) \in [0,\infty)\times\mathbb{R}^d$, where $a(t,x) = (a^{i,j}(t,x))$ is a non-negative definite symmetric $d\times d$-matrix and $b(t,x) = (b^i(t,x))$ is a d-dimensional vector, $\partial_i = \frac{\partial}{\partial x_i}, \nabla = (\partial_1, ..., \partial_d)$ the gradient operator with $\cdot$ standing for the inner product on $\mathbb{R}^d$, and $\nu(t,x,dz)$ is a Lévy kernel, namely, $\forall (t,x)\in[0,\infty)\times\mathbb{R}^d, \nu(t,x,\cdot)$ is a σ-finite measure on $(\mathbb{R}^d\setminus\{0\}, \mathcal{B}(\mathbb{R}^d\setminus\{0\}))$ satisfying that

$$\int_{\mathbb{R}^d\setminus\{0\}} \frac{|z|^2}{1+|z|^2}\nu(t,x,dz) < \infty.$$

The Markov process associated with such a generator L can be determined either as a solution to the martingale problem for L as well as the Dirichlet form approach or as a solution to a (jump type) SDE related to L. The latter is particularly more useful in both analytic and constructive aspects.

Let us present a way to construct the associated SDE. We start with a standard probability set-up $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t\in[0,\infty)})$. Given a σ-finite measure space $(U, \mathcal{B}(U), \lambda)$. Usually, $(U, \mathcal{B}(U), \lambda)$ is a parameter space measuring jumps of stochastic processes to be considered. A canonical example for this space is that $(U, \mathcal{B}(U), \lambda) = (\mathbb{R}^d\setminus\{0\}, \mathcal{B}(\mathbb{R}^d\setminus\{0\}), \lambda)$. Following, e.g. [25] (cf. Theorem I.8.1), one can construct a canonical Poisson random measure $N(dt, dy, \omega)$ on $[0,\infty)\times U$, i.e.

$$N : \mathcal{B}([0,\infty))\times\mathcal{B}(U)\times\Omega \to \mathbb{N}\cup\{0\}\cup\{\infty\}$$

with intensity measure λ:

$$\mathbf{E}(N([0,t), A, \cdot) = t\lambda(A), \quad \forall t\geq 0, \quad \forall A\in\mathcal{B}(U) \text{ with } \lambda(A) < \infty.$$

In dealing with Lévy generators, without loss of generality, one usually works with L having the following more convenient form

$$\begin{aligned} Lf(t,x) &= \frac{1}{2}\sum_{i,j=1}^{d} a^{i,j}(t,x)\partial_i\partial_j f(t,x) + \sum_{i=1}^{d} b^i(t,x)\partial_i f(t,x) \\ &+ \int_{\mathbb{R}^d\setminus\{0\}} \{f(t,x+z) - f(t,x) - \frac{z1_{\{|z|<1\}}\cdot\nabla f(t,x)}{1+|z|^2}\}\nu(t,x,dz), \end{aligned} \tag{2.2}$$

by changing the drift vector b appropriately. So from now on, let us take the Lévy generator L with this expression. For such L, one can choose a $d\times m$-matrix $\sigma(t,x) = (\sigma^{i,j}(t,x))$ and a d-dimensional vector $c : [0,\infty)\times\mathbb{R}^d\times U \to \mathbb{R}^d$ such that $\sigma(t,x)\sigma^T(t,x) = a(t,x)$ and

$$\int_U 1_A(c(t,x,y))\lambda(dy) = \int_{\mathbb{R}^d\setminus\{0\}} 1_A(z)\nu(t,x,dz), \tag{2.3}$$

hold $\forall(t,x)\in[0,\infty)\times\mathbb{R}^d$ and $\forall A\in\mathcal{B}(\mathbb{R}^d\setminus\{0\})$. This bridge equality is extremely important since it allows modelling with Lévy type processes to be linked to Lévy processes with jumps. Effectively from having to consider a Lévy kernel ν, by this equality we can simply consider the Lévy measure λ. Now this λ is required to satisfy the property of Lévy measures.

Furthermore, let $\{W_t\}_{t\in[0,\infty)}$ be an m-dimensional $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-Brownian motion. Then, a jump SDE associated with L can be formulated as follows (cf. [17, 48])

$$dS_t = b(t,S_t)dt + \sigma(t,S_t)dW_t + \int_U c(t,S_{t-},y)\tilde{N}(dt,dy), \tag{2.4}$$

where $\tilde{N}$ is the compensating $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-martingale measure of N

$$\tilde{N}(dt,dy,\omega) := N(dt,dy,\omega) - dt\lambda(dy). \tag{2.5}$$

Moreover, under certain usual assumptions on the diffusion matrix a, the drift vector b, and the assumption that Lévy kernel ν with generalized polar decomposition (cf. [48]), there exists a unique (pathwise) solution to the above jump type SDE which is the jump Markov process generated by L. We call such a jump Markov process *a Lévy-type process* associated with the generator Lévy L.

Let us end this section by presenting two examples of the Lévy kernel ν (time independent) fulfilling all the assumptions of M. Tsuchiya [48] thus each of them generates a unique jump Markov process. Such processes are called stable like processes.

Example I. Let $\alpha : \mathbb{R}^d \longrightarrow (0, 2)$ be measurable and set

$$\nu(t, x, dz) := \frac{dz}{|z|^{d+\alpha(x)}}$$

the Lévy generator L with this kernel ν was considered in [5].

Example II. Let $\alpha \in (0, 2)$ and set

$$\nu(t, x, dz) := \frac{\mu(x, d\theta)d\rho}{\rho^{1+\alpha}}$$

where $(\rho, \theta) \in (0, \infty) \times \mathbb{S}^{d-1}$ stands for the polar coordinate of $z \in \mathbb{R}^d \setminus \{0\}$, $\mu(x, \cdot)$ is a finite Borel measure on the sphere $\mathbb{S}^{d-1}$ and μ is measurable on $x \in \mathbb{R}^d$. Stable like jump-diffusions generated by the Lévy generator L with such kernel ν has been studied intensively in [32].

2.2 The Stochastic control problem and the derivation of the Hamiltonian

We start with a controlled jump Markov process $S_t = S_t^{(u)}$ over a time interval $[0, T]$ for an arbitrarily fixed $0 < T < \infty$,which is given by

$$
\begin{aligned}
dS_t &= \mu(t, S_t, u_t)dt + \sigma(t, S_t, u_t)dW_t \\
&+ \int_{U \setminus U_0} c_1(t, S_t, u_t, z)\tilde{N}(dt, dz) + \int_{U_0} c_2(t, S_t, u_t, z)N(dt, dz).
\end{aligned} \tag{2.6}
$$

Let S_t be an $\mathbb{R}^d$ measurable function which represents the price of an asset over an interval $[0, T]$. Let $\mu : [0, T] \times \mathbb{R}^d \times \mathcal{Q} \rightarrow \mathbb{R}^d$ be the drift term (the average rate of growth of the asset), $\sigma : [0, T] \times \mathbb{R}^d \times \mathcal{Q} \rightarrow \mathbb{R}^{d \otimes m}$ be the volatility of the underlying asset, both measurable functions and let $W_t = (W_t^1, ..., W_t^m)$ be an m-dimensional Brownian motion and the integral kernels $c_1, c_2 : [0, T] \times \mathbb{R}^d \times \mathcal{Q} \times U \rightarrow \mathbb{R}^d$.

There are a few remarks that we should make regarding the above jump-type SDE. Firstly, note that the above is split into two sections the the first being the first two terms which is referred to as the drift and the diffusion, implying the price of the asset can change by drifting and diffusing. The second section is the remaining integrals which are referred to as the jump parts, which handles the jump changes in the asset price.

A required condition on the coefficients of the jump term is that these are locally bounded and integrable functions.

Let the control process $u_t = u(t, \omega), (t, \omega) \in [0, \infty) \times \Omega$, taking values in a given Borel set $\mathcal{Q} \in \mathcal{B}(\mathbb{R}^d)$, is assumed to be predictable and càdlàg(continuous to right with limits to the left). Its value is chosen at any instant $t \in [0, T]$ with the aim to control the process S_t. We say that the control process u is admissible and write $u \in \mathcal{A}$ if there exists a unique and strong solution $S_t = S_t^{(u)}, t \in [0, T]$.

Let $\mathcal{A}$ denote the set of all admissible controls and let the performance criterion be defined in its general form

$$J(u) = \mathbb{E}\left(\int_0^T f(t, S_t, u_t)dt + g(S_T)\right), \quad u \in \mathcal{A}. \tag{2.7}$$

If $u \in \mathcal{A}$, and $S_t = S_t^{(u)}$ is the corresponding solution, then this solution is deemed as being admissible if the following condition is satisfied

$$\mathbb{E}[\int_0^T |f(t, S_t, u_t)|dt + max(0, g(S_{T-}))] < \infty, \tag{2.8}$$

for $f : [0,T] \times \mathbb{R}^d \times \mathcal{Q} \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ both given bounded continuous functions.

The objective of the stochastic control problem is to achieve a maximum for $J(u)$ over all of $u \in \mathcal{A}$, in other words, we want to find a $\hat{u} \in \mathcal{A}$ such that

$$J(\hat{u}) = \sup_{u \in \mathcal{A}} J(u). \tag{2.9}$$

Such a $\hat{u}$ is referred to as being the optimal control of the system. Furthermore, if $\hat{S}_t = S_t^{(\hat{u})}$ is the solution to the jump-type SDE (2.6) corresponding to $\hat{u}$, then the pair $(\hat{S}, \hat{u})$ is called the optimal pair. Let us now move to deriving the associated generator such that it can be used to construct the Hamiltonian when considering Dynamic programming.

To derive the generator, we make use of Itô's formula and the formulation of the martingale problem. The formulation of the martingale problem is as follows, $\forall f \in \mathcal{C}_b^2(\mathbb{R}^d)$,

$$f(S_t) - f(S_0) - \int_0^t (Af)(S_s)ds,$$

is a martingale then A is the generator of S_t, subsequently we can obtain the Hamiltonian.

Let start with the integral form of (2.6) with the aim to obtain an expression for the jump Markov process S_t, thus we get

$$
\begin{aligned}
S_t &= S_0 + \int_0^t \mu(s, S_s, u_s)ds + \int_0^t \sigma(s, S_s, u_s)dW_s \\
&+ \int_0^t \int_{U\setminus U_0} c_1(s-, S_{s-}, u_{s-}, z)\tilde{N}(ds, dz) \\
&+ \int_0^t \int_{U_0} c_2(s-, S_{s-}, u_{s-}, z)N(ds, dz)
\end{aligned}
$$

The next step is to employ the general form of Itô's formula, this to hold $\forall f \in \mathcal{C}^2(\mathbb{R}^d)$. So in application we obtain

$$
\begin{aligned}
f(S_t) - f(S_0) &= \int_0^t [(\frac{\partial f}{\partial s})(S_s)]\mu(s, S_s, u_s)ds + \int_0^t [(\frac{\partial f}{\partial s})(S_s)]\sigma(s, S_s, u_s)dW_s \\
&+ \frac{1}{2}\int_0^t [(\frac{\partial^2 f}{\partial s^2})(S_s)]\sigma(s, S_s, u_s)ds \\
&+ \int_0^{t^+} \int_{U\setminus U_0} [f(S_{s-} + c_1(s-, S_{s-}, u_{s-}, z)) \\
&\qquad -f(S_{s-})]\tilde{N}(ds, dz) \\
&+ \int_0^{t^+} \int_{U_0} [f(S_{s-} + c_2(s-, S_{s-}, u_{s-}, z)) \\
&\qquad -f(S_{s-})]N(ds, dz) \\
&+ \int_0^t \int_{U\setminus U_0} [f(S_{s-} + c_1(s-, S_{s-}, u_{s-}, z)) - f(S_{s-}) \\
&\qquad -c_1(s-, S_{s-}, u_{s-}, z)(\frac{\partial f}{\partial s})(S_{s-})]ds\lambda(dz).
\end{aligned}
$$

So with the aim being to rearrange into the martingale problem form, let

$$
\begin{aligned}
f(S_t) - f(S_0) &= \int_0^t [(\frac{\partial f}{\partial s})(S_s)]\mu(s, S_s, u_s)ds + \int_0^t [(\frac{\partial f}{\partial s})(S_s)]\sigma(S, S_s, u_s)dW_s \\
&+ \frac{1}{2}\int_0^t [(\frac{\partial^2 f}{\partial s^2})(S_s)]\sigma(s, S_s, u_s)\sigma^T(s, S_s, u_s)ds \\
&+ \int_0^{t^+}\int_{U\backslash U_0} [f(S_{s-} + c_1(s-, S_{s-}, u_{s-}, z)) - f(S_{s-})]\tilde{N}(ds, dz) \\
&+ \int_0^{t^+}\int_{U_0} [f(S_{s-} + c_2(s-, S_{s-}, u_{s-}, z)) - f(S_{s-})]\tilde{N}(ds, dz) \\
&+ \int_0^{t^+}\int_{U_0} [f(S_{s-} + c_2(s-, S_{s-}, u_{s-}, z)) - f(S_{s-})]ds\lambda(dz) \\
&+ \int_0^{t}\int_{U\backslash U_0} [f(S_{s-} + c_1(s-, S_{s-}, u_{s-}, z)) - f(S_{s-}) \\
&\qquad -c_1(s-, S_{s-}, u_{s-}, z)(\frac{\partial f}{\partial s})(S_{s-})]ds\lambda(dz).
\end{aligned}
$$

By collecting like terms we get

$$
\begin{aligned}
f(S_t) - f(S_0) &= \int_0^t [(\frac{\partial f}{\partial s})(S_s)]\mu(s, S_s, u_s)ds + \int_0^t [(\frac{\partial f}{\partial s})(S_s)]\sigma(s, S_s, u_s)dW_s \\
&+\frac{1}{2}\int_0^t [(\frac{\partial^2 f}{\partial s^2})(S_s)]\sigma(s, S_s, u_s)\sigma^T(s, S_s, u_s)ds \\
&+\int_0^{t^+} [\int_{U\backslash U_0} f(S_{s-} + c_1(s-, S_{s-}, u_{s-}, z)) - f(S_{s-}) \\
&+\int_{U_0} f(S_{s-} + c_2(s-, S_{s-}, u_{s-}, z)) - f(S_{s-})]\tilde{N}(ds, dz) \\
&+\int_0^{t} [\int_{U\backslash U_0} f(S_{s-} + c_1(s-, S_{s-}, u_{s-}, z)) - f(S_{s-}) \\
&\qquad -c_1(s-, S_{s-}, u_{s-}, z)(\frac{\partial f}{\partial s})(S_{s-}) \\
&+\int_{U_0} f(S_{s-} + c_2(s-, S_{s-}, u_{s-}, z)) - f(S_{s-})]\lambda(dz)ds.
\end{aligned}
$$

Finally it can be seen that we can rearrange this into the form of the martingale problem and hence be able obtain the generator.

$$
\begin{aligned}
& f(S_t) - f(S_0) - \int_0^{t+} [[(\frac{\partial f}{\partial s})(S_s)]\mu(s, S_s, u_s) \\
& + \frac{1}{2}[(\frac{\partial^2 f}{\partial s^2})(S_s)]\sigma(s, S_s, u_s)\sigma^T(s, S_s, u_s) \\
& + (\int_{U\setminus U_0} f(S_{s-} + c_1(s-, S_{s-}, u_{s-}, z)) - f(S_{s-}) \\
& - c_1(s-, S_{s-}, u_{s-}, z)(\frac{\partial f}{\partial s})(S_{s-}) \\
& + \int_{U_0} f(S_{s-} + c_2(s-, S_{s-}, u_{s-}, z)) - f(S_{s-}))\lambda(dz)]ds \\
= & \\
& \int_0^{t+} ([(\frac{\partial f}{\partial s})(S_s)]\sigma(s, S_s, u_s)dW_s \\
& + [\int_{U\setminus U_0} f(S_{s-} + \eta_1(s-, S_{s_-}, u_{s-}, z)) - f(S_{s-}) \\
& + \int_{U_0} f(S_{s-} + c_2(s-, S_{s-}, u_{s-}, z)) - f(S_{s-})]\tilde{N}(ds, dz).
\end{aligned}
$$

This is in the required form of the martingale problem since on the left hand side we have by the condition of the martingale problem, a local martingale. Note that when taking the expectation of a martingale, you get zero as a result, that is if the martingale is zero at $t = 0$. On the right hand side we have the additional terms which are themselves martingales - the first being Brownian motion and the second being a jump martingale. As a result we obtain the generator, for $f \in C_0^2(\mathbb{R}^d)$ by

$$
\begin{aligned}
Hf(x) &= (\frac{\partial f}{\partial s})(x)\mu(t,x,u) + \frac{1}{2}(\frac{\partial^2 f}{\partial s^2})(x)\sigma(t,x,u)\sigma^T(t,x,u) \\
&+[\int_{U\backslash U_0} f(x + c_1(t,x,u,z)) - f(x) - c_1(t,x,u,z)(\frac{\partial f}{\partial s})(x) \\
&+ \int_{U_0} f(x + c_2(t,x,u,z)) - f(x)]\lambda(dz).
\end{aligned}
\tag{2.10}
$$

2.3 The relation to Dynamic Programming

In this section we will aim to make a connection between the stochastic control problem and the Dynamic programming technique, what we need is to establish a procedure such that we end up with a maximum for the value function.

We start with a controlled jump Markov process S_t represented by

$$
\begin{aligned}
dS_t &= \mu(t, S_t, u_t)dt + \sigma(t, S_t, u_t)dW_t \\
&+ \int_{U\backslash U_0} c_1(t, S_t, u_t, z)\tilde{N}(dt, dz) + \int_{U_0} c_2(t, S_t, u_t, z)N(dt, dz)
\end{aligned}
$$

with the associated generator (2.10), however by using this formulation and by incorporating 4 adapted processes, we define the Hamiltonian as

$A : [0, T] \times \mathbb{R}^d \times \mathcal{Q} \times \mathbb{R}^d \times \mathbb{R}^{d\otimes m} \times \mathcal{R} \to \mathbb{R}$ via

$$
\begin{aligned}
& A(t,r,u,p,q,m^{(1)},m^{(2)}) \\
= \; & f(t,r,u)+\mu(t,r,u)p+\frac{1}{2}\sigma^T(t,r,u)q \\
& +\int_{U\backslash U_0} m^{(1)}(t,z)c_1(t,r,u,z)\lambda(dz) \qquad (2.11)\\
& +\int_{U_0}[m^{(2)}(t,z)c_2(t,r,u,z)+c_2(t,r,u,z)p]\lambda(dz),
\end{aligned}
$$

where $\mathcal{R}$ is the collection of those $\mathbb{R}^{d\otimes d}$-valued processes $m:[0,\infty)\times U\times\Omega\to\mathbb{R}^{d\otimes d}$ such that the integrals in the above formulation converge absolutely.

It is known by [40], that the adjoint equation for the adapted processes $p(t)\in\mathbb{R}^d, q(t)\in\mathbb{R}^{d\otimes m}$ and $m^{(1)}(t,z), m^{(2)}(t,z)\in\mathbb{R}^{d\otimes d}$ corresponding to an admissible pair (S,u) is the following backward stochastic differential equation

$$
\begin{aligned}
dp(t) \;=\; & -\nabla_r A(t,S_t,u_t,p(t),q(t),m^{(1)}(t,\cdot),m^{(2)}(t,\cdot))dt \\
& +q(t)dW_t+\int_{U\backslash U_0} m^{(1)}(t-,z)\tilde{N}(dt,dz) \\
& +\int_{U_0} m^{(2)}(t-,z)N(dt,dz), \qquad (2.12)
\end{aligned}
$$

with terminal condition

$$
p(T)=\nabla g(S_T)\,. \qquad (2.13)
$$

Assuming that A is differentiable with respect to r. Let us state now the verification result into our setting by following the lines of [19]

Theorem 2.3.1 *Let $(\hat{S},\hat{u})$ be an admissable pair and let $Diag(\chi)$ represent the characteristics of the jump size(taking the value 0 or 1 along the diagonal)* *over U. Suppose that there exists an $\{\mathcal{F}_t\}$-adapted solution $(\hat{p}(t), \hat{q}(t), \hat{m}(t,z))$ of the corresponding adjoint equation such that for $u \in \mathcal{A}$*

$$\mathbb{E}[\int_0^T (\hat{S}_t - S_t^{(u)})^T \{\hat{q}(t)\hat{q}(t)^T + \int_U [\hat{m}(t,z) Diag(\chi(z)) \hat{m}(t,z)^T \lambda(dz)]\} \times (\hat{S}_t - S^{(u)}(t))dt] < \infty, \tag{2.14}$$

$$\mathbb{E}[\int_0^T \hat{p}^T(t) \{\int_U [c(t, S_{t-}, u_t, z) Diag(\chi(z)) c^T(t, S_{t-}, u_t, z)\lambda(dz)] + \sigma(t, S_t, u_t)\sigma^T(t, S_t, u_t)\}\hat{p}(t)dt] < \infty, \tag{2.15}$$

and suppose further that for all $t \in [0, T]$

$$A(t, \hat{S}_t, \hat{u}_t, \hat{p}(t), \hat{q}(t), \hat{m}(t,.)) = \sup_{u \in \mathcal{A}} A(t, \hat{S}_t, u_t, \hat{p}(t), \hat{q}(t), \hat{m}(t,.)) \,. \tag{2.16}$$

If

$$\hat{A}(r) := max_{u \in \mathcal{A}} H(t, r, u, \hat{p}(t), \hat{q}(t), \hat{m}(t, \cdot)), \quad (Arrow\ condition) \tag{2.17}$$

exists and is a concave function of r, then $(\hat{S}, \hat{u})$ is an optimal pair

Remark 2.3.2 *For (2.17) to hold it suffices that the function*

$$(r, u) \to A(t, r, u, \hat{p}(t), \hat{q}(t), \hat{m}(t, \cdot))$$

is concave, for all $t \in [0, T]$.

It is well known that there is indeed a relation between both the ideas when considering the diffusion case alone. But our intention is to demonstrate this can be done when jumps are involved. By studying the Hamiltonian, the intention is to express parts of our formulation in the form of 4 adapted processes each relating to a particular expression within the Hamiltonian.

Remark 2.3.3 *It should be noted here that the adjoint process $m^{(1)}(t, \cdot), m^{(2)}(t, \cdot)$ represents the jumps of the r-gradient of our value function V(t,r).*

The aim is to prove the formulas (2.24) - (2.27) that we have for the adapted processes, but firstly we are required set the stochastic control problem within a Markovian framework such that it would be suitable for Dynamic Programming. So let

$S_t = S(t) = S_t^{s,r}$ be the solution of our model for $t \geq s$ and initially let $S(s) = r$, so

$$J_u(s,r) = \mathbb{E}[\int_s^T f(t, S_t^{s,r}, u(t))dt + g(S^{s,r}(T))], \qquad u \in \mathcal{A}. \tag{2.18}$$

Let the value function be defined

$$V(s,r) = \sup_{u \in \mathcal{A}} J_u(s,r). \tag{2.19}$$

We assume that an optimal Markovian control $u^*(t,r) = u^*$ exists for our problem and let $S^*(t)$ be the corresponding optimal state process (for example $S^*(t)$ is the solution when $u = u^*(t, S(t))$), this being a Markov control. Now, under certain conditions we will assume the following Hamilton-Jacobi-Bellman(HJB) equation of dynamic programming holds,

$$\sup_{u \in \mathcal{A}} A^u(t,r) = \sup_{u \in \mathcal{A}} A(t,r,u) = A(t,r,u^*(t,r)) = 0. \tag{2.20}$$

Let the corresponding hamiltonian A be defined similarly to (2.10)

$$
\begin{aligned}
&A(t,r,u)\\
&= f(t,r,u) + \frac{\partial V}{\partial t}(t,r) + \sum_{i=1}^{d} \mu_i(t,r,u)\frac{\partial V}{\partial r_i}(t,r)\\
&+\frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^T)_{ij}(t,r,u)\frac{\partial^2 V}{\partial r_i \partial r_j}(t,r)\\
&+\{\int_{U\setminus U_0} V(t,r+c_1(t,r,u,z)) - V(t,r)\\
&-\sum_{i=1}^{d} c_i(t,r,u,z).\frac{\partial V}{\partial r_i}(t,r)\\
&+\int_{U_0} V(t,r+c_2(t,r,u,z)) - V(t,r)\}\lambda(dz).
\end{aligned}
$$

The next step is to differentiate $A(t,r,u^*(t,r))$ with respect to r_h and evaluate the result at the point $r = S^*(t)$, and we obtain

$$
\begin{aligned}
0 \;&= \frac{\partial f}{\partial r_h}(t,S^*(t),u^*(t,S^*(t))) + \frac{\partial^2 V}{\partial t \partial r_h}(t,S^*(t)\\
&+\sum_{i=1}^{d}\mu_i(t,S^*(t),u^*(t,S^*(t))).\frac{\partial^2 V}{\partial r_i \partial r_h}(t,S^*(t))\\
&+\sum_{i=1}^{d}\frac{\partial \mu_i}{\partial r_h}(t,S^*(t),u^*(t,S^*(t))).\frac{\partial V}{\partial r_i}(t,S^*(t))\\
&+\frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^T)_{ij}(t,S^*(t),u^*(t,S^*(t))).\frac{\partial^3 V}{\partial r_i \partial r_j \partial r_h}(t,S^*(t))\\
&+\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial(\sigma\sigma^T)_{ij}}{\partial r_h}(t,S^*(t),u^*(t,S^*(t))).\frac{\partial^2 V}{\partial r_i \partial r_j}(t,S^*(t))
\end{aligned}
$$

$$
\begin{aligned}
&+\{\int_{U\setminus U_0}\sum_{i=1}^{d}\frac{\partial V}{\partial r_i}(t,S^*(t)+c_1(t,S^*(t),u^*(t,S^*(t)),z))\\
&\quad\times(\delta_{ih}+\frac{\partial c_i}{\partial r_h}(t,S^*(t),u^*(t,S^*(t)),z))\\
&-\sum_{i=1}^{d}c_i(t,S^*(t),u^*(t,S^*(t)),z).\frac{\partial^2 V}{\partial r_i\partial r_h}(t,S^*(t))\\
&-\frac{\partial c_i}{\partial r_h}(t,S^*(t),u^*(t,S^*(t)),z).\frac{\partial V}{\partial r_i}(t,S^*(t))\\
&-\frac{\partial V}{\partial r_h}(t,S^*(t))\\
&+\int_{U_0}\frac{\partial V}{\partial r_i}(t,S^*(t)+c_2(t,S^*(t),u^*(t,S^*(t)),z))\\
&\quad\times(\delta_{ih}+\frac{\partial c_i}{\partial r_h}(t,S^*(t),u^*(t,S^*(t)),z))\\
&-\frac{\partial V}{\partial r_h}(t,S^*(t))\}\lambda(dz),
\end{aligned}
$$

where

$$
\begin{aligned}
\delta_{ih}&=1 \qquad\qquad \text{if } i=h,\\
\delta_{ih}&=0 \qquad\qquad \text{if } i\neq h.
\end{aligned}
$$

We must notice that the terms containing the derivatives of $A(t,r,u)$ with respect to u would vanish when u is optimal i.e $u=u^*$, this is since the hamiltonian is maximal at this point. Now let us choose

$$
\gamma_h(t)=\frac{\partial V}{\partial r_h}(t,S^*(t)). \qquad\qquad h=1,...,d. \tag{2.21}
$$

By Itô's formula a slightly different form than what was used previously as now we have time and spatial dependance to consider, we have

$$
\begin{aligned}
d\gamma_h(t) \;&= \sum_{i=1}^{d} \frac{\partial^2 V}{\partial r_i \partial r_h}(t, S^*(t))(\mu_i(t, S^*(t), u^*(t))dt + \sigma_i(t, S^*(t), u^*(t))dW(t)) \\
&+ \frac{\partial^2 V}{\partial t \partial r_h}(t, S^*(t))dt \\
&+ \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^T)_{ij}(t, S^*(t), u^*(t, S^*(t))).\frac{\partial^3 V}{\partial r_i \partial r_j \partial r_h}(t, S^*(t))dt \\
&+ \int_{U\setminus U_0} \{\frac{\partial V}{\partial r_h}(t, S^*(t-) + c_1(t, S^*(t), u^*(t, S^*(t)), z)) \\
&\qquad - \frac{\partial V}{\partial r_h}(t, S^*(t-)) \\
&- \sum_{i=1}^{d} \frac{\partial^2 V}{\partial r_i \partial r_h}(t, S^*(t))c_i(t, S^*(t), u^*(t, S^*(t)), z)\}\lambda(dz)dt \\
&+ \int_{U\setminus U_0} [\frac{\partial V}{\partial r_h}(t, S^*(t-) + c_1(t, S^*(t), u^*(t, S^*(t)), z)) \\
&\qquad - \frac{\partial V}{\partial r_h}(t, S^*(t-))]\tilde{N}(dt, dz) \\
&+ \int_{U_0} [\frac{\partial V}{\partial r_h}(t, S^*(t-) + c_2(t, S^*(t), u^*(t, S^*(t)), z)) \\
&\qquad - \frac{\partial V}{\partial r_h}(t, S^*(t))]N(dt, dz).
\end{aligned}
$$

The next step now is to substitute into the above what we found for the value of $\frac{\partial^2 V}{\partial t \partial r_h}$ from one of our previous calculations, and thus we obtain,

$$
\begin{aligned}
d\gamma_h(t) \;&= -[\frac{\partial f}{\partial r_h} + \sum_{i=1}^{d} \mu_i \frac{\partial^2 V}{\partial r_i \partial r_h} + \sum_{i=1}^{d} \frac{\partial \mu_i}{\partial r_h}\frac{\partial V}{\partial r_i} \\
&+ \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^T)_{ij}\frac{\partial^3 V}{\partial r_i \partial r_j \partial r_h} + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial(\sigma\sigma^T)_{ij}}{\partial r_h}\frac{\partial^2 V}{\partial r_i \partial r_j} \\
&+ \sum_{i=1}^{d} \int_{U\setminus U_0} \{(\sum_{i=1}^{d}(\frac{\partial V}{\partial r_i}(t, r + c_1) - \frac{\partial V}{\partial r_i})\frac{\partial c_i}{\partial r_h}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad (\frac{\partial V}{\partial r_h}(t, r+c_1) - \frac{\partial V}{\partial r_h}) - c_i \frac{\partial^2 V}{\partial r_i \partial r_h}) \\
&+ \quad \int_{U_0} \frac{\partial V}{\partial r_i}(t, r+c_2)\frac{\partial c_i}{\partial r_h} + (\frac{\partial V}{\partial r_h}(t, r+c_2) - \frac{\partial V}{\partial r_h})\}\lambda(dz)]dt \\
&+ \quad \sum_{i=1}^{d} \frac{\partial^2 V}{\partial r_i \partial r_h}(\mu_i dt + \sigma_i dW(t)) + \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^T)_{ij}\frac{\partial^3 V}{\partial r_i \partial r_j \partial r_h}dt \\
&+ \quad \int_{U \setminus U_0} \{(\frac{\partial V}{\partial r_h}(t, r+c_1) - \frac{\partial V}{\partial r_h}) - \sum_{i,j=1}^{d} \frac{\partial^2 V}{\partial r_i \partial r_h}.c_i \\
&+ \quad \int_{U_0} (\frac{\partial V}{\partial r_h}(t, r+c_2) - \frac{\partial V}{\partial r_h})\}\lambda(dz)dt \\
&\quad + \int_{U \setminus U_0} (\frac{\partial V}{\partial r_h}(t, r+c_1) - \frac{\partial V}{\partial r_h})\tilde{N}(dt, dz) \\
&+ \quad \int_{U_0} (\frac{\partial V}{\partial r_h}(t, r+c_2) - \frac{\partial V}{\partial r_h})N(dt, dz).
\end{aligned}
$$

With like terms cancelling and some manipulation the above simplifies to

$$
\begin{aligned}
d\gamma_h(t) \quad &= -[\frac{\partial f}{\partial r_h} + \sum_{i=1}^{d} \frac{\partial \mu_i}{\partial r_h}.\frac{\partial V}{\partial r_i} + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial(\sigma\sigma^T)_{ij}}{\partial r_h}\frac{\partial^2 V}{\partial r_i \partial r_j} \\
&+ \int_{U \setminus U_0} \{\sum_{i=1}^{d}(\frac{\partial V}{\partial r_i}(t, r+c_1) - \frac{\partial V}{\partial r_i})\frac{\partial c_i}{\partial r_h} \\
&+ \int_{U_0} (\frac{\partial V}{\partial r_i}(t, r+c_2)\frac{\partial c_i}{\partial r_h})\}\lambda(dz)]dt \\
&+ \int_{U \setminus U_0} (\frac{\partial V}{\partial r_h}(t, r+c_1) - \frac{\partial V}{\partial r_h})\tilde{N}(dt, dz) \\
&+ \int_{U_0} (\frac{\partial V}{\partial r_h}(t, r+c_2) - \frac{\partial V}{\partial r_h})N(dt, dz) \\
&+ \sum_{i=1}^{d}\sum_{j=1}^{m} \frac{\partial^2 V}{\partial r_i \partial r_j}\sigma_{ij}dW_j.
\end{aligned}
$$

Therefore we obtain the hamiltonian as in (2.11)

$$A(t,r,u,p,q,m^{(1)},m^{(2)}) = f(t,r,u) + \mu(t,r,u).p + \frac{1}{2}\sigma^T(t,r,u).q$$
$$\int_{U\setminus U_0} m^{(1)}(t,z).c_1(t,r,u,z)\lambda(dz)$$
$$+\int_{U_0}\{m^{(2)}(t,z)c_2(t,r,u,z) + c_2(t,r,u,z).p\}\lambda(dz). \tag{2.22}$$

Moreover if we let

$$r(t) = S^*(t), u(t) = u^*(t, S^*(t-))$$

and take $p_i(t), q_{jk}(t), m^{(1)}(t,z)$,$m^{(2)}(t,z)$ as in (2.24) - (2.27), we get the following adjoint equation (a Backward Stochastic Differential equation).

$$d\gamma_h(t) = -\frac{\partial A}{\partial r_h}(t, r(t), u(t), p(t), q(t), m^{(1)}(t,\cdot), m^{(2)}(t,\cdot))dt + q_h(t)dW_t$$
$$+\int_{U\setminus U_0} m^{(1)}(t,z)\tilde{N}(dt,dz) + \int_{U_0} m^{(2)}(t,z)N(dt,dz). \tag{2.23}$$

Thus we can summarize by means of the following theorem.

Theorem 2.3.4 *Assume that the value function $V(s,r) \in C^{1,3}(\mathbb{R}\times\mathbb{R}^d)$ (defined in (2.19))and that there exists an optimal Markov control $u^*(t,r)$ for problem (2.19) with corresponding solution $S^*(t)$ of (2.6). Then, the processes $p(t)$,$q(t)$ and $m^{(1)}(t,\cdot), m^{(2)}(t,\cdot)$ given by*

$$p_i(t) = \frac{\partial V}{\partial r_i}(t, S^*(t)), \tag{2.24}$$

$$q_{ik}(t) = \sum_{j=1}^{d} \sigma_{jk}(t, S^*(t), u^*(t))\frac{\partial V}{\partial r_i \partial r_j}(t, S^*(t)), \tag{2.25}$$

$$m^{(1)}(t,z) = \frac{\partial V}{\partial r_i}(t, S^*(t) + c_1(t, S^*(t), u^*(t), z)) - \frac{\partial V}{\partial r_i}(t, S^*(t)). \tag{2.26}$$

$$m^{(2)}(t,z) = \frac{\partial V}{\partial r_i}(t, S^*(t) + c_2(t, S^*(t), u^*(t), z)) - \frac{\partial V}{\partial r_i}(t, S^*(t)). \tag{2.27}$$

for all $i = 1,\ldots,d$, $j = 1,\ldots,d$, $k = 1,\ldots,m$, solve the adjoint equation (2.23).

2.4 Applications to finance

In this section we will consider two examples which arises from financial optimization. Let us consider a financial market model consisting of 2 investment possibilities:

(i) a risk free security(i.e a bond), where the price $S_0(t)$ is

$$dS_0(t) = \rho_t S_0(t)dt, \qquad S_0(0) > 0, \tag{2.28}$$

where ρ_t is a locally bounded deterministic function.

(ii) a risky security(i.e a stock), where the price $S_1(t)$ is

$$\begin{aligned} dS_1(t) &= S_1(t-)[\mu_t dt + \sigma_t dW(t) \\ &+ \int_{U\setminus U_0} c_1(t-,z)\tilde{N}(dt,dz) + \int_{U_0} c_2(t-,z)N(dt,dz)], \end{aligned} \tag{2.29}$$

where μ, σ are deterministic bounded functions,where $\mu_t > \rho_t$, $\sigma_t \neq 0$ and $c_1$and c_2 are locally bounded. To ensure that $S_1(t) > 0, \forall t$, we assume that $c_1(t,z), c_2(t,z) \geq -1$ a.a t, z.

This example may be regarded as an extension of the Black-Scholes market.

A portfolio is a predictable process $\theta(t) = (\theta_0(t), \theta_1(t)) \in \mathbb{R}^2$ giving the number of units held at any time t of the risk free and risky security. Now the corresponding wealth process $X(t) = X^{(\theta)}(t)$ is given by

$$X(t) = \theta_0(t)S_0(t) + \theta_1(t)S_1(t), \qquad t \in [0, T]. \tag{2.30}$$

The portfolio is self-financing if

$$X(t) = X(0) + \int_0^t \theta_0(s)dS_(s) + \int_0^t \theta_1(s)dS_1(s). \tag{2.31}$$

Let $u(t) = \theta_1(t)S_1(t)$ denote the amount invested in the risky asset. Now by combining (2.28) and (2.29) into (2.31) we obtain

$$\begin{aligned}
&X(t) = X(0) + \int_0^t \theta_0(s)\rho_s S_0(s)ds \\
&+ \int_0^t \theta_1(s-)\{S_1(s-)[\mu_s ds + \sigma_s dW(s) + \int_{U\backslash U_0} c_1(t,z)\tilde{N}(dt,dz) \\
&+ \int_{U_0} c_2(t,z)N(dt,dz)]\},
\end{aligned}$$

now making use of the $u(t)$ substitution above and differentiating we have

$$\begin{aligned}
&dX(t) = \rho_t\theta_0(t)S_0(t)dt + u(t)\mu_t dt + u(t)\sigma_t dW(t) \\
&+u(t-)\int_{U\backslash U_0} c_1(t,z)\tilde{N}(dt,dz) + u(t-)\int_{U_0} c_2(t,z)N(dt,dz),
\end{aligned}$$

now by substituting (2.30), we get

$$\begin{aligned}
&dX(t) = \{\rho_t X(t) + (\mu_t - \rho_t)u(t)\}dt + \sigma_t u(t)dW(t) \\
&+u(t-)\int_{U\backslash U_0} c_1(t,z)\tilde{N}(dt,dz) + u(t-)\int_{U_0} c_2(t,z)N(dt,dz),
\end{aligned} \tag{2.32}$$

where $X(0) \in \mathbb{R}$ is given. We call the control $u(t)$ admissible and write $u(t) \in \mathcal{A}$ if (2.32) has a unique solution $X(t) = X^{(u)}(t)$ such that $\mathbb{E}[(X^{(u)}(t))^2] < \infty$. The control u is called tame if the corresponding wealth process (2.32) is square integrable over $[0, T] \times \Omega$. Such a requirement is used to exclude doubling strategies that would gain arbitrary profit at time T, but with consequence of unbounded intermediate losses.

Example 1 Mean-Variance Portfolio Selection.

The objective is to find the admissible portfolio $u(t)$ such that it minimizes the variance. We know that

$$Var[X(T)] = \mathbb{E}[(X(T) - \mathbb{E}[X(T))^2],$$

under the condition that $\mathbb{E}[X(T)] = A$,A is a given constant.

By the Lagrange multiplier method, the problem can be reduced to a problem of simply minimizing for a given $a \in \mathbb{R}$,

$$\mathbb{E}[(X(T) - a)^2],$$

without any constraint. If we consider the expression

$$\begin{aligned}\mathbb{E}[(X(T) - A)^2 - \lambda([X(T)] - A)] &= \mathbb{E}[X^2(T) - 2(A + \tfrac{\lambda}{2})X(T) + A^2 + \lambda A] \\ &= \mathbb{E}[(X(T) - (A + \tfrac{\lambda}{2}))^2] + \tfrac{\lambda^2}{4}, \qquad \lambda \in \mathbb{R},\end{aligned}$$

and consider the problem

$$\sup_{\omega \in \mathcal{A}} \mathbb{E}[-\frac{1}{2}(X^{(u)}(T) - a)^2], \tag{2.33}$$

where $X(t) = X^{(u)}(t)$ is given by (2.32) and the set $\mathcal{A}$ of admissible strategies consists of the predictable tame portfolios $u(t)$ such that (2.32) has a strong solution over [0,T]. The solution to this problem for an Itô diffusion is already well known, so our intention is to illustrate the solution using the ideas of the maximum principle. The Hamiltonian will take the form

$$
\begin{aligned}
&A(t, x, u, p, q, m^{(1)}, m^{(2)}) = \{\rho_t x + (\mu_t - \rho_t)u\}p + \sigma_t u q \\
&\quad + u \int_{U \setminus U_0} c_1(t, z) m^{(1)}(t, z) \lambda(dz) \\
&+ u \int_{U_0} \{c_2(t, z) m^{(2)}(t, z) + c_2(t, z) p\} \lambda(dz),
\end{aligned}
\tag{2.34}
$$

and the associated adjoint equation with terminal condition will take the form

$$
\begin{aligned}
&dp(t) = -\rho_t p(t) dt + q(t) dW(t) \\
&+ \int_{U \setminus U_0} m^{(1)}(t, z) \tilde{N}(dt, dz) + \int_{U_0} m^{(2)}(t, z) N(dt, dz),
\end{aligned}
\tag{2.35}
$$

$$
p(T) = -X(T) + a. \tag{2.36}
$$

Now to solve this we will choose a process of the form $p(t) = \phi_t X(t) + \psi_t$, where ϕ_t, ψ_t are deterministic functions. Now using (2.32) and differentiating the given process $p(t)$ we get

$$
\begin{aligned}
dp(t) &= \phi_t\{\rho_t X(t) - (\mu_t - \rho_t)u(t)\}dt + \phi_t\sigma_t u(t)dW(t) \\
&+\phi_t u(t-)\int_{U\setminus U_0} c_1(t,z)\tilde{N}(dt,dz) + \phi_t u(t-)\int_{U_0} c_2(t,z)N(dt,dz) \\
&+X(t)\phi_t' dt + \psi_t' dt \\
&= \\
&[\phi_t\rho_t X(t) + \phi_t(\mu_t - \rho_t)u(t) + X(t)\phi_t' + \psi_t']dt \\
&\phi_t u(t-)\int_{U\setminus U_0} c_1(t,z)\tilde{N}(dt,dz) + \phi_t u(t-)\int_{U_0} c_2(t,z)N(dt,dz) \\
&+\sigma_t\phi_t u(t)dW(t).
\end{aligned}
\tag{2.37}
$$

Now c.f coefficients of (2.35) and (2.37), we obtain

$$
(dt) : \phi_t\rho_t X(t) + \phi_t(\tilde{\mu_t} - \rho_t)u(t) + X(t)\phi_t' + \psi_t' = -\rho_t(\phi_t X(t) + \psi_t), \tag{2.38}
$$

$$
(dW(t)) : \phi_t\sigma_t u(t) = q(t), \tag{2.39}
$$

$$
(\tilde{N}(dt,dz)) : \phi_t u(t)c_1(t,z) = m^{(1)}(t,z), \tag{2.40}
$$

$$
(N(dt,dz)) : \phi_t u(t)c_2(t,z) = m^{(2)}(t,z). \tag{2.41}
$$

Let $\hat{u} \in \mathcal{A}$ be a candidate for an optimal control and let $\hat{X}(t)$ be the corresponding wealth process with corresponding solution $(\hat{p}(t), \hat{q}(t), \hat{m}^{(1)}(t,z), \hat{m}^{(2)}(t,z))$ of the adjoint eqn (2.35) and (2.36). Then, evaluating the Hamitlonian with these said parameters yields

$$
\begin{aligned}
&A(t,\hat{X}(t),u,\hat{p}(t),\hat{q}(t),\hat{m}^{(1)}(t,z),\hat{m}^{(2)}(t,z)) = \rho_t\hat{X}(t)\hat{p}(t) \\
&+u[(\mu_t - \rho_t)\hat{p}(t) + \sigma_t\hat{q}(t) + \{\int_{U\setminus U_0} c_1(t,z)\hat{m}^{(1)}(t,z) \\
&+\int_{U_0} c_2(t,z)\hat{m}^{(2)}(t,z) + c_2(t,z)\hat{p}(t)\}\lambda(dz)].
\end{aligned}
$$

this being a linear expression in u, we can make a natural guess that the coefficient u vanishes due to the maximum principle hence leading to

$$(\mu_t - \rho_t)\hat{p}(t) + \sigma_t \hat{q}(t) + \{\int_{U\setminus U_0} c_1(t,z)\hat{m}^{(1)}(t,z)$$

$$\int_{U_0} c_2(t,z)\hat{m}^{(2)}(t,z) + c_2(t,z)\hat{p}(t)\}\lambda(dz) = 0. \tag{2.42}$$

The next step if we substitute (2.39),(2.40)and (2.41) into (2.42) we obtain an expression for the optimal control $\hat{u(t)}$, so

$$\hat{u(t)} = \frac{(\rho_t - (\mu_t + \int_{U_0} c_2(t,z)\lambda(dz))\hat{p}(t)}{\sigma_t^2\phi_t + \{\int_{U\setminus U_0} c_1(t,z)^2\phi_t + \int_{U_0} c_2(t,z)^2\phi_t\}\lambda(dz)},$$

which equals after substituting

$$\hat{p}(t) = \phi_t \hat{X}(t) + \psi_t$$

and letting

$$\gamma_t = \sigma_t^2 + \{\int_{U\setminus U_0} c_1(t,z)^2 + \int_{U_0} c_2(t,z)^2\}\lambda(dz),$$

and

$$\tilde{\mu_t} = \mu_t + \int_{U_0} c_2(t,z)\lambda(dz),$$

we get

$$\hat{u(t)} = \frac{(\rho_t - \tilde{\mu_t})(\phi_t\hat{X}(t) + \psi_t)}{\phi_t\gamma_t}. \tag{2.43}$$

Now if we make $u(t)$ the subject in (2.38) we get an alternate expression. So

$$\hat{u}(t) = \frac{[(\phi_t\rho_t + \phi_t')\hat{X}(t) + \rho_t(\phi_t\hat{X}(t) + \psi_t) + \psi_t']}{\phi_t(\rho_t - \tilde{\mu_t})}, \tag{2.44}$$

the next step now is to compare the like terms in the aim to find expressions for ϕ_t and ψ_t.

$$(\hat{X}(t)) : (\rho_t - \tilde{\mu_t})^2\phi_t - [2\rho_t\phi_t + \phi_t']\gamma_t = 0 \qquad ; \phi_T = -1.$$
$$(freeterms) : (\rho_t - \tilde{\mu_t})^2\psi_t - [\rho_t\psi_t + \psi_t']\gamma_t = 0 \qquad ; \psi_T = a. \quad (2.45)$$

and finally we can deduce the following

$$\phi_t = -\exp[\int_t^T \{\frac{(\rho_s - \tilde{\mu_s})^2}{\gamma_s} - 2\rho_s\}ds], \qquad t \in [0, T]. \quad (2.46)$$

$$\psi_t = a\exp[\int_t^T \{\frac{(\rho_s - \tilde{\mu_s})^2}{\gamma_s} - \rho_s\}], \qquad t \in [0, T]. \quad (2.47)$$

With this choice of ϕ_t and ψ_t the processes (2.38)-(2.41) solve the adjoint equation and by (2.42) we see that all conditions of the sufficient maximum principle are satisfied. So from this example we can conclude,

Theorem 2.4.1 *For $\hat{u}(t)$ given by Equation (2.43), this is the optimal control leading to an optimal solution of the mean-variance portfolio selection problem, when X obeys (2.32).*

Example 2 Consumption-portfolio optimization problem

In this problem we consider the consumption-portfolio optimization with almost sure terminal condition in a financial market model. The situation is that we allow the agent to withdraw consumption from his wealth. The wealth process then satisfies the following equation

$$\begin{aligned} dR(t) &= \{\rho_t R(t) + (b(t) - \rho_t)u(t) - \theta(t)\}dt + \sigma(t)u(t)dW(t) \quad (2.48) \\ &\quad + u(t-)\int_{U\setminus U_0} c_1(t,z)\tilde{N}(dt,dz) + u(t-)\int_{U_0} c_2(t,z)N(dt,dz). \end{aligned}$$

Our objective is to solve the following consumption-portfolio optimization problem:

$$\sup_{(\theta,u)\in\mathcal{A}} \mathbb{E}[\int_0^T \exp(-\int_0^t \delta(s)ds)[\frac{\theta(t)^\gamma}{\gamma}]dt], \tag{2.49}$$

subject to an almost sure terminal wealth constraint,

$$R(T) \geq 0, \quad a.s. \tag{2.50}$$

for $\mathcal{A}$ to be the set of predictable consumption-portfolio pairs $d = (\theta, u)$ with u being tame and θ being nonnegative, such that (2.48) has a strong solution over $[0,T]$ as well as (2.50) is satisfied. Where in the expression (2.49), $\delta : [0,T] \to \mathbb{R}$ is a given bounded deterministic function and $\gamma < 1$ is a given nonzero constant. The dynamic programming approach is not an appropriate method to solve this type of stochastic control problem with an almost sure terminal condition like (2.50). The sufficient maximum principle outlined previously is better since it allows us to apply constraints. So let us take the following terminal condition,

$$0 \geq \mathbb{E}[(\hat{R}(T) - R(T))\hat{p}(T)]. \tag{2.51}$$

Remark 2.4.2 *For more discussion regarding to the transversality condition (2.51), the reader is referred to Theorem 2.1 in [40].*

By considering $R-\bar{r}$ instead of R, where $\bar{r}$ is the nonzero minimal terminal wealth coefficient, the Hamiltonian we considered is of the following form

$$\begin{aligned} & A(t,r,\theta,u,p,q,m^{(1)},m^{(2)}) \\ = \; & \exp(-\int_0^t \delta(r)dr)(\frac{\theta^\gamma}{\gamma}) - p\theta + \rho p x \\ & + u\{p(b-\rho) + q\sigma + \int_{U\setminus U_0} m^{(1)}(t-,z)c_1(t,z)\lambda(dz) \\ & + \int_{U_0} \{m^{(2)}(t-,z)c_2(t,z) + c_2(t,z)p\}\lambda(dz) \,. \end{aligned} \tag{2.52}$$

On the other hand, the modified adjoint equation now becomes

$$\begin{aligned} dp(t) &= -\rho(t)p(t)dt + q(t)dW(t) + \int_{U\backslash U_0} m^{(1)}(t-,z)d\tilde{N}(dt,dz) \\ &+ \int_{U_0} m^{(2)}(t-,z)dN(dt,dz), \end{aligned} \tag{2.53}$$

with

$$\mathbb{E}[(\hat{R}(T) - R(T))\hat{p}(T)] \leq 0. \tag{2.54}$$

Now let the pair $(\hat{\theta}, \hat{u}) \in \mathcal{A}$ have corresponding solution $\hat{R}$ and $(\hat{p}, \hat{q}, \hat{m}^{(1)}, \hat{m}^{(2)})$ of equations (2.48) and (2.53).

The value of θ which maximizes $A(t, \hat{R}(r), \hat{u}, \hat{p}(t), \hat{q}(t), \hat{m}^{(1)}(t,\cdot), \hat{m}^{(2)}(t,\cdot))$ is given by

$$\theta = \hat{\theta}(t) = [\exp(\int_0^t \delta(s)ds)\hat{p}(t)]^{\frac{1}{(\gamma-1)}}. \tag{2.55}$$

Since the Hamiltonian A contains u in a linear form, it is natural to assume that the coefficient of u will vanish due to the maximum principle. Hence we have the following

$$\begin{aligned} \hat{p}(t)(b(t) &- \rho_t) + \sigma(t)\hat{q}(t) + \int_{U\backslash U_0} c_1(t-,z)\hat{m}^{(1)}(t,z)\lambda(dz) \\ &+ \int_{U_0} \{c_2(t-,z)\hat{m}^{(2)}(t,z) + c_2(t-,z)\hat{p}(t)\}\lambda(dz) = 0. \end{aligned} \tag{2.56}$$

If we assume that it is optimal to consume at a rate proportional to the current wealth $\hat{S}(t)$ then

$$\hat{p}(t) = f(t)\hat{R}(t)^{(\gamma-1)}, \tag{2.57}$$

for some deterministic differentiable function f. Now differentiating (2.57) and applying Itô's formula, we get

$$\begin{aligned}
d\hat{p}(t) &= f'(t)\hat{R}^{\gamma-1}dt + (\gamma-1)f(t)\hat{R}(t)^{\gamma-2}d\hat{R}(t) \\
&\quad + \frac{1}{2}(\gamma-1)(\gamma-2)f(t)\hat{R}(t)^{\gamma-3}\sigma_t^2\hat{u}(t)^2 dt \\
&\quad + \int_{U\backslash U_0} f(t)\{(\hat{R}(t)+c_1(t,z)\hat{u}(t))^{\gamma-1} - \hat{R}(t)^{\gamma-1} \\
&\quad -(\gamma-1)\hat{R}(t)^{\gamma-2}\hat{u}(t)c_1(t,z)\}\lambda(dz)dt \\
&\quad + \int_{U\backslash U_0} f(t)\{(\hat{R}(t)+c_1(t,z)\hat{u}(t))^{\gamma-1} - \hat{R}(t)^{\gamma-1}\}\tilde{N}(dt,dz) \\
&\quad + \int_{U_0} f(t)\{(\hat{R}(t)+c_2(t,z)\hat{u}(t))^{\gamma-1} - \hat{R}(t)^{\gamma-1}\}N(dt,dz).
\end{aligned}$$

Moreover, by substituting (2.48) into the above and then comparing like coefficients with those from the adjoint equation (2.53), we obtain the following

$$\hat{m}^{(1)}(t,z) = f(t)\hat{R}(t)^{\gamma-1}\{(1+c_1(t,z)\hat{u}(t)\hat{R}(t)^{-1})^{\gamma-1} - 1\}. \tag{2.58}$$

$$\hat{m}^{(2)}(t,z) = f(t)\hat{R}(t)^{\gamma-1}\{(1+c_2(t,z)\hat{u}(t)\hat{X}(t)^{-1})^{\gamma-1} - 1\}. \tag{2.59}$$

$$\hat{q}(t) = (\gamma-1)f(t)\sigma(t)\hat{u}(t)\hat{R}(t)^{\gamma-2}. \tag{2.60}$$

and

$$f'(t) + \alpha_t f(t) + (1-\gamma)\exp\{\int_0^t [\frac{\delta(s)}{(\gamma-1)}]ds\} f(t)^{\frac{\gamma}{(\gamma-1)}} = 0, \tag{2.61}$$

where α_t is obtained by simply collecting the common expression $f(t)\hat{R}(t)^{\gamma-1}$ and placing it oustide the bracket and then multiplying by $\hat{R}(t)^{1-\gamma}$ to leave simply the $f(t)$ as the dominant term. We have

$$\begin{aligned}
\alpha_t &= \gamma\rho_t + (\gamma-1)(b(t)-\rho_t)\hat{u}(t)\hat{R}(t)^{-1} \\
&\quad + \frac{1}{2}(\gamma-1)(\gamma-2)\sigma^2(t)\hat{u}(t)^2\hat{R}(t)^{-2} \\
&\quad + \int_{U\backslash U_0} \{(1+c_1(t,z)\hat{u}(t)^{-1})^{\gamma-1} \\
&\qquad -1-(\gamma-1)\hat{u}(t)\hat{R}(t)^{-1}c_1(t,z)\}\lambda(dz).
\end{aligned} \tag{2.62}$$

The next step is to substitute (2.58), (2.59) and (2.60) into (2.56), but before we do that we are going to rearrange it slightly as follows

$$
\begin{aligned}
&((b(t)+\int_{U_0} c_2(t,z)\lambda(dz))-\rho_t)\hat{p}(t)+\sigma(t)\hat{q}(t)\\
&\quad+\int_{U\backslash U_0} c_1(t,z)\hat{m}^{(1)}(t,z)\lambda(dz)+\int_{U_0} c_2(t,z)\hat{m}^{(2)}(t,z)\}\lambda(dz)=0.
\end{aligned}
$$

For simplicity now we let

$$
\tilde{b(t)} := b(t)+\int_{U_0} c_2(t,z)\lambda(dz),
$$

and after this substitution we have

$$
\begin{aligned}
&(\tilde{b}(t)-\rho_t)\hat{p}(t)+\sigma(t)^2\hat{u}(t)f(t)(\gamma-1)\hat{R}(t)^{\gamma-2}\\
&+\{\int_{U\backslash U_0} c_1(t,z)\{(1+c_1(t,z)\hat{u}(t)\hat{R}(t)^{-1})^{\gamma-1}-1 \qquad (2.63)\\
&+\int_{U_0} c_2(t,z)\{(1+c_2(t,z)\hat{u}(t)\hat{R}(t)^{-1})^{\gamma-1}-1\}\}\lambda(dz)=0.
\end{aligned}
$$

$$(2.64)$$

Thus we derive the relation

$$
F(\hat{u}(t)\hat{R}(t)^{-1})=0,
$$

where

$$
\begin{aligned}
F(\pi) \; := \; & \tilde{b}(t)-\rho_t+\sigma^2(t)(\gamma-1)\pi\\
&+\{\int_{U\backslash U_0} c_1(t,z)\{(1+c_1(t,z)\pi)^{\gamma-1}-1\}\\
&+\int_{U_0} c_2(t,z)\{(1+c_2(t,z)\pi)^{\gamma-1}-1\}\}\lambda(dz).
\end{aligned}
$$

On the other hand, note that $F(0)=\tilde{b}_t-\rho_t>0$. Therefore, there exists $\hat{\pi}(t)>0$ such that

$$
F(\hat{\pi}(t))=0, \qquad (2.65)
$$

since $F(\pi) < 0$ for large π. Let us take

$$\hat{u}(t)\hat{R}(t)^{-1} = \hat{\pi}(t), \tag{2.66}$$

and α_t to be as in (2.62). To solve Equation (2.61) we are required to make a change of variable

$$h(t) = [\exp(\int_0^t \delta(s)ds)f(t)]^{\frac{1}{(1-\gamma)}}.$$

Utilizing this change of variable, we get

$$\begin{aligned} f(t) &= \exp(-\int_0^t \delta(r)dr)\Big\{[f(T)^{\frac{1}{(1-\gamma)}}]\{\exp\int_0^T [\frac{\delta(s)}{(1-\gamma)}]ds\} \\ &\quad \times\{\exp\int_t^T [\frac{(\alpha_r - \delta(r))}{(1-\gamma)}]dr\} \\ &\quad + \int_t^T \exp\{-\int_s^t [\frac{(\alpha_r - \delta(r))}{(1-\gamma)}]dr\Big\} ds\}^{1-\gamma}, \end{aligned} \tag{2.67}$$

which solves Equation (2.61). Thus, by using (2.66),(2.55) and (2.57), we derive the following expression for $\hat{\theta}$

$$\hat{\theta}(t) = \{\exp\int_0^t [\frac{\delta(s)}{(\gamma-1)}]ds\}f(t)^{\frac{1}{(\gamma-1)}}\hat{R}(t). \tag{2.68}$$

Moreover, the corresponding Equation (2.48) then becomes

$$\begin{aligned} d\hat{R}(t) &= \hat{R}(t)\{[\rho_t + (\tilde{b}(t) - \rho_t)\hat{\pi}(t) - \{\exp\int_0^t [\frac{\delta(s)}{(\gamma-1)}]ds\}f(t)^{\frac{1}{(\gamma-1)}}]dt \\ &\quad +\sigma_t\hat{\pi}(t)dW(t) + \hat{\pi}(t-)\int_{U\setminus U_0} c_1(t,z)\tilde{N}(dt,dz) \\ &\quad +\hat{\pi}(t-)\int_{U_0} c_2(t,z)N(dt,dz)\}, \end{aligned}$$

with solution being given by

$$
\begin{aligned}
& \hat{R}(t) \\
= \; & \hat{R}(0)\exp\{\int_0^t [\rho_s + (\tilde{b}(s) - \rho_s)\hat{\pi}(s) \\
& -\{\exp\int_0^s [\frac{\delta(r)}{(\gamma-1)}]dr\} f(s)^{\frac{1}{(\gamma-1)}} - \frac{1}{2}\hat{\pi}(s)\sigma(s)^2]ds \\
& + \int_0^t \sigma(s)\hat{\pi}(s)dW(s) + \int_0^t (\int_{U\backslash U_0} \log(1 + c_1(t,s))N(ds,dz) \qquad (2.69) \\
& - \int_{U\backslash U_0} c_1(s,z)\lambda(dz))ds + \int_0^t \int_{U_0} c_2(t,z)N(ds,dz)\}.
\end{aligned}
$$

From this it is reasonable to think that the optimal wealth process will satisfy the terminal condition with equality, as excess wealth is worthless. Although to achieve this we must have as a result of (2.69), f(T) = 0, which in turn gives us, by (2.67)

$$
f(t) = \exp(-\int_0^t \delta(s)ds)[\int_t^T \exp\{-\int_s^t [\frac{\alpha_r - \delta(r)}{(1-\gamma)}]dr\}]^{1-\gamma}. \qquad (2.70)
$$

Hence we have

$$
f(s) \sim (T-s)^{1-\gamma}, \qquad as \quad s \to T-.
$$

Therefore

$$
\int^T f(s)^{\frac{1}{(\gamma-1)}} ds \sim \int^T (T-s)^{-1} ds = \infty
$$

which by (2.69) gives that $\hat{R}(T) = 0$, as required.

With $\hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{m}^{(1)}(t,\cdot), \hat{m}^{(2)}(t,\cdot), f(t)$ as they are in Equations (2.66), (2.57), (2.60), (2.58), (2.59) and (2.70), respectively, all the conditions of the maximum principle are satisfied, including the terminal condition (2.54).

Theorem 2.4.3 *An optimal control $d = (\theta, u)$ for problem (2.49) subject to (2.48) is given when θ satisfies (2.68) and u satisfies (2.66) with $f(t)$ satisfying (2.70).*

These two applications will be revisited in chapter 4, however the coefficients of the jump term will be explicitely defined, where the models will be driven by a stable-like process.

Chapter 3

An optimal control problem associated with SDEs driven by Lévy-type processes

In this chapter we consider the optimal control of a Lévy type driven financial market consisting of a pair of processes, a wealth process and the average past consumption process. We aim to control these using 3 parameter processes, the cumulative consumption up to time t, the fraction of the wealth that the investor chooses to invest in the risky asset and a reflection type control representing any additional income. We consider a risky asset with the dynamics of a geometric Lévy type process. This type of problem has been considered in depth in [7] - [9], and more recently in [26]. The aim is to follow their pattern oultined but within our new setting.

3.1 Optimal control problem and the results

We start with the same preamble as in the previous section 2.1. Let us take a compensated Poisson random measure (2.5) with Lévy measure $\lambda(dz)$. Let Z_t be a Lévy type process defined by the Lévy-Itô decomposition

$$Z_t = \mu t + \int_0^t \theta(s)dW_s + \int_0^t \int_{U\setminus U_0} c_1(z)\tilde{N}(ds,dz) + \int_0^t \int_{U_0} c_2(z)N(ds,dz),$$

where μ is a constant, $\theta : [0,T] \to \mathbb{R}$ and $c_1, c_2 : U \to \mathbb{R}$ are measurable, and $U_0 \in \mathcal{B}(U)$ fulfilling $\lambda(U_0) < \infty$ is arbitrarily fixed. Throughout the chapter, we assume that

$$\int_{U_0} (e^{c_2(z)} - 1)\lambda(dz) < \infty. \tag{3.1}$$

Starting from this, the jump type SDE we are concerned is formulated in the following manner

$$dS_t = b(t, S_t)dt + \sigma(t, S_{t-})dZ_t,$$

where $b : [0,T] \times \mathbb{R} \to \mathbb{R}, \sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ are measurable.

By allowing $b(t,S_t) = b(t)S_t$ and $\sigma(t,S_t) = \sigma(t)S_t$, i.e., the coefficients b, σ are linear on S_t, we get a geometric Lévy-type process $S_t = S_0 e^{Z_t}$ with initial data $S_0 > 0$.

By the Itô formula (cf. e.g. Theorem II.5.1 in [25]), we get the following equation for S_t

$$\begin{aligned} dS_t &= b(t)S_t dt + \frac{1}{2}\sigma(t)^2 S_t dt + \sigma(t)S_t dW_t \\ &+ S_t \int_{U\setminus U_0} (e^{c_1(z)} - 1 - c_1(z))\lambda(dz)dt \\ &+ S_{t-} \int_{U\setminus U_0} (e^{c_1(z)} - 1)\tilde{N}(dt,dz) + S_{t-} \int_{U_0} (e^{c_2(z)} - 1)N(dt,dz). \end{aligned} \tag{3.2}$$

Remark 3.1.1 *An alternative formulation of Equation (3.2) is as follows. From our assumption (3.1), we can use drift transformation*

$$\hat{b}(t) = b(t) + \frac{1}{2}\sigma(t)^2 + \int_U (e^{c(z)} - 1 - c(z)1_{\{U\setminus U_0\}}(z))\lambda(dz),$$

to get the following equation for S_t

$$dS_t = \hat{b}(t)S_t dt + \sigma(t)S_t dW_t + S_{t-}\int_U (e^{c(z)} - 1)\tilde{N}(dt, dz). \tag{3.3}$$

However, with a stable-like process being a special case, we will consider the jump type SDE (3.2) in its formulation with the jumps being separate entities.

Remark 3.1.2 *By applying the bridge equality (2.3) and fixing* (t, S_t)*, we get a bimeasurable bijection such that* $c : [0,\infty) \times \mathbb{R} \times U \to \mathbb{R}$*. This idea is going to be used throughout.*

Let D_β be defined by

$$D_\beta := \{(x,y); y > 0, y + \beta x > 0\}$$

where the lower boundaries of D_β are the lines $y = 0$ and $y + \beta x = 0$, and the proportionlity constant $\beta > 0$ is used to describe the damping rate of the average past consumption. This means the bigger β refers to a preference to more recent past consumption by the investor.

Now based on the processes Z_t and S_t, we aim to discuss and construct the wealth process $X = X_t^x$ and the average past consumption process $Y = Y_t^y$. Both X and Y are adapted. But before we do the construction let us firstly describe what each represents. It will also be seen that this pair of processes will be dependent on specific parameter processes, namely (π_t, G_t, L_t). It is these that will represent the control, it is assumed that these controls are admissible if they satisfy the following conditions:

1. $G_t = \int_0^t g_s ds$, and $t \mapsto g_t$ is a non-decreasing adapted càdlàg process of finite variation such that $0 \leq g_t \leq M_1$ for all $t \geq 0$, for some $M_1 > 0$, and that $g_t > 0$ only for t when $X_t \geq 0$.

2. L_t is a non-decreasing adapted cádlág process such that $L_{0-} = 0, L_t \geq 0$ a.s., $\mathbb{E}[L_t] < \infty$ for all $t \geq 0$, $\Delta L_t > 0$ only for such t that $X_{t-} \in D_\beta$ and $X_{t-} + \Delta X_t \notin D_\beta$ and $L_t^c > 0$ only for such t that $X_t \leq 0$. Let L_t^c denote the continuous part of L_t.

3. π_t is an adapted càdlàg process with values in [0,1].

4. π_t, G_t, L_t are processes such that if $(x, y) \in D_\beta$, then $(X_t, Y_t) \in \bar{D}_\beta$ a.s. holds for $t \geq 0$.

Let us now briefly mention what each process represents and how they all relate within our financial model. So we already know X_t represents the wealth process(the amount of money owned). Let G_t denote the cumulative consumption up to time t, and $\pi_t \in [0, 1]$ the fraction of the investers money he/she decides to invest in the risky asset(i.e stock), this being subject to S_t, respectively. Let $r \geq 0$ represent the interest rate of a safe asset(i.e a bond).

It must be realized that the control G_t is only increasing when the investor has a non-negative wealth $(X_t \geq 0)$(i.e only spend money when there is money to spend). On the other hand, the process L_t is a control to adjust the wealth should it become negative, this corresponds to a situation where there is some sort of additional income (i.e selling an asset). This control is only used when a debt is incurred($X_t \leq 0$), and its jump part is only used when (X_t, Y_t) might exit D_β.

The process Y_t represents the average past consumption process(i.e ave. amount of money spent), which must be greater than 0. The ideas behind (3.4) and (3.5) is that should a jump take the process (X_t, Y_t) out of $\bar{D}_\beta$, then the admissible control will bring it back immediately. Note that $\bar{D}_\beta$ is the closure of D_β.

Now for the construction, if we consider a self-financing investment policy according to the portfolio π_t:

$$\frac{dX_t}{X_{t-}} = (1-\pi_t)\frac{dB_t}{B_t} + \pi_t \frac{dS_t}{S_{t-}},$$

where B_t denotes the riskless bond given by $dB_t = rB_t dt$, we can construct X_t as follows

$$\begin{aligned}
X_t &= x - G_t + \int_0^t \sigma(s)\pi_s X_s dW_s + L_t \\
&+ \int_0^t (r + ([b(s) + \frac{1}{2}\sigma(s)^2 + \int_{U\setminus U_0} (e^{c_1(z)} - 1 - c_1(z))\lambda(dz)] - r)\pi_s) X_s ds \\
&+ \int_0^t \pi_{s-} X_{s-} \int_{U\setminus U_0} (e^{c_1(z)} - 1)\tilde{N}(ds, dz) \\
&+ \int_0^t \pi_{s-} X_{s-} \int_{U_0} (e^{c_2(z)} - 1) N(ds, dz).
\end{aligned} \tag{3.4}$$

The process Y_t can be constructed from $dY_t = -\beta Y_t dt + \beta dG_t$, namely

$$Y_t = ye^{-\beta t} + \beta \int_0^t e^{-\beta(t-s)} dG_s. \tag{3.5}$$

Remark 3.1.3 *No transaction costs are considered with these controls.*

For our model we require the control to be Markovian, so allow the parameters (π_t, g_t, L_t) to be fixed and allow the value to be determined by the value of the pair (X_t, Y_t). The objective is to maximize the expected utility. The investor will derive the utility from this average past consumption Y_t rather than from the present consumption this is following the line of [9].

So if we let the value function be defined by

$$v(x,y) = \sup_{(\pi.,g.,L.)\in\mathcal{A}} \mathbb{E}^{(X^{(\pi.,g.,L.)},Y^{(\pi.,g.,L.)})}[\int_0^\infty e^{-\alpha s}u(Y_s)ds], \qquad (3.6)$$

where $X_t^{(\pi.,g.,L.)}, Y_t^{(\pi.,g.,L.)}$ are the processes X_t, Y_t given $(\pi., g., L.)$, and $\alpha > 0$ is the damping rate of the utility. Our goal now is to characterize v as a viscosity solution to the Hamilton-Jacobi-Bellman(HJB) equation, to be given below (3.8).

The utility function $u(\cdot)$ is assumed to be concave on $[0,\infty]$, differentiable implying that it is continuous and locally bounded and also strictly increasing. Gossen's law is also adhered to, implying the dependance on the consumption rate and the dependance on the hasty investor maximizing the utility. i.e maximizing their satisfaction.

Remark 3.1.4 *By [9],it can be shown that an optimal control $(\pi^*, g^*, L^*) \in \mathcal{A}$ exists, such that*

$$v(x,y) = \mathbb{E}^{(X^{(\pi^*,g^*,L^*)},Y^{(\pi^*,g^*,L^*)})}[\int_0^\infty e^{-\alpha s}u(Y_s^*)ds]$$

holds. In the sequel, the trajectory associated with this optimal control is denoted by (X_t^*, Y_t^*).

Let us now define the generator A that is associated with the pair (X_t, Y_t)

$$\begin{aligned}
Av(x,y) &= -\alpha v - \beta y v_y + \sigma(t)\pi x v_{xx} \\
&+\{(r + \pi([b(t) + \frac{1}{2}\sigma(t)^2 + \int_{U\setminus U_0}(e^{c_1(z)} - 1 - c_1(z))\lambda(dz)] - r))xv_x \\
&+ \int_{U\setminus U_0}(v(x + \pi x(e^{c_1(z)} - 1), y) - v(x,y) - \pi x v_x(e^{c_1(z)} - 1))\lambda(dz) \\
&+ \int_{U_0}(v(x + \pi x(e^{c_2(z)} - 1), y) - v(x,y))\lambda(dz)\} \\
&+u(y) - y(v_x - \beta v_y), \qquad \pi \in [0,1] \qquad (3.7)
\end{aligned}$$

Further , we set

$$Nv = v_x.1_{\{x\leq 0\}},$$

and

$$Mv = (\beta v_y - v_x).1_{\{x\geq 0\}},$$

where v_x, v_{xx}, v_y denote the partial derivatives of order 1 or 2 with respect to x and y. Let the operators M and N correspond to the continuous parts of the controls G_t, L_t in (3.4) and (3.5).

In [7]-[9], the authors characterized this value function as a constrained viscosity soultion of the associated Hamilton-Jacobi-Bellman(HJB) equation, we will proceed in a similar fashion. For more detail on this HJB equation refer to these studies.

So the HJB equation (integro-variational inequality) which we aim to solve is defined

$$\begin{aligned}
&\max\{Nv, \sup_{\pi,g\in\mathcal{A}}\{Av\}, Mv\} = 0 \quad in\ D_\beta, \\
&v = 0 \quad outside\ of\ D_\beta.
\end{aligned} \tag{3.8}$$

Now a useful equality that we will utilize throughout the proofs of the existence and uniqueness of the solution to (3.8) is

$$Av(x,y) = F((x,y), v, v_x, v_y, v_{xx}; v, v_x, v),$$

where F is defined as follows

$$\begin{aligned}
& F((x,y), w, s, q, m; \phi, p, \phi) \\
= \;& -\alpha w - \beta y q + \sigma(t)\pi x m \\
& + \max_{0\leq\pi\leq 1}\{(r + \pi([b(t) + \frac{1}{2}\sigma(t)^2 \\
& \quad + \int_{U\setminus U_0}(e^{c_1(z)} - 1 - c_1(z))\lambda(dz)] - r))xs \\
& \quad + B_1^\pi((x,y),\phi,p) + B_2^\pi((x,y),\phi,0)\} + u(y) - y(s - \beta q),
\end{aligned} \tag{3.9}$$

with s, m, p, q being all scalars and

$$
\begin{aligned}
& B_1^\pi((x,y),\phi,p) \\
:= & \int_{U\setminus U_0} (\phi(x+\pi x(e^{c_1(z)}-1),y)-\phi(x,y)-\pi x p(e^{c_1(z)}-1))\lambda(dz). \\
& B_2^\pi((x,y),\phi,0) \\
:= & \int_{U_0} (\phi(x+\pi x(e^{c_2(z)}-1),y)-\phi(x,y))\lambda(dz).
\end{aligned}
$$

where

$$B^\pi((x,y),\phi) := B_1^\pi((x,y),\phi,p) + B_2^\pi((x,y),\phi,0).$$

Let the function space for our generator A be $Q(\bar{D}_\beta)$. We will not define this space explicitly instead the specific requirements will be stated. As a result of the perturbation term caused by the Brownian motion, which is an elliptic differential operator, $Q(\bar{D}_\beta)$ is required to be a function space consisting of bounded and integrable functions which are twice differentiable for the variables x, y. Notice that convergence is not a worry due to the property of Lévy measures

$$\int_U \frac{|z|^2}{1+|z|^2}\lambda(dz) < \infty.$$

If it holds for $\tilde{v} \in C^2(\bar{D}_\beta) \cap Q(\bar{D}_\beta)$ that

$$N\tilde{v} \le 0, \quad M\tilde{v} \le 0, \ and \ \sup_{\pi, g \in \mathcal{A}} A\tilde{v} \le 0 \, in \, D_\beta,$$

then it is well known that, for the value function v,

$$v \le \tilde{v} \ \ in \, D_\beta.$$

For a proof of this refer to [26].

Definition 3.1.5 *(see [7] and [8]). Let $E \subset \bar{D}_\beta$.*

(i) Any $v \in C(\bar{D}_\beta)$ is a viscosity subsolution (resp. supersolution) of (3.8) in E iff for all $(x,y) \in E$ and all $\phi \in C^2(\bar{D}_\beta) \cap Q(\bar{D}_\beta)$ such that (x,y) is a global maximizer (resp. minimizer) of $v - \phi$ relative to E, it holds that

$$\max(N\phi, \sup_g(F(\cdot, v, \phi_x, \phi_y, \phi_{xx}; \phi, \phi_x, \phi)), M\phi)(x,y) \geq 0$$

and

$$(resp. \max(N\phi, \sup_g(F(\cdot, v, \phi_x, \phi_y, \phi_{xx}; \phi, \phi_x, \phi)), M\phi)(x,y) \leq 0).$$

(ii) $v \in C(\bar{D}_\beta)$ is a constrained viscosity solution of (3.8) iff v is a viscosity subsolution of (3.8) in $\bar{D}_\beta$ and a supersolution of (3.8) in D_β.

Definition 3.1.6 *Let $E \subset \bar{D}_\beta$. Any $v \in C(\bar{D}_\beta)$ is a strict supersolution in E iff for every $(x,y) \in E, \phi \in C^2(\bar{D}_\beta) \cap Q(\bar{D}_\beta)$ such that (x,y) is a global maximizer of $v - \phi$ relative to E, there exists $v > 0$ such that*

$$\max(N\phi, \sup_g(F(\cdot, v, \phi_x, \phi_y, \phi_{xx}; \phi, \phi_x, \phi)), M\phi)(x,y) \leq -v.$$

Now we are in place to give our main results that deal with the existence and uniqueness of the viscosity solution.

Theorem 3.1.7 *The well defined and bounded value function v(x,y) is a constrained viscosity solution of (3.8).*

Theorem 3.1.8 *For each $\gamma > 0, \rho \geq 0$ choose $\alpha > 0$ as in (3.6) so that $\alpha > k(\gamma, \rho)$, where we define a finite constant $k(\gamma, \rho)$ by*

$$\begin{aligned} & k(\gamma,\rho) \\ = \ & \max[\gamma(r + \pi([b(t) + \frac{1}{2}\sigma(t)^2 + \int_{U\setminus U_0} (e^{c_1(z)} - 1 - c_1(z))\lambda(dz)] - r)) \\ & + \sigma(t)\pi\rho + \int_{U\setminus U_0} [(1 + \pi(e^{c_1(z)} - 1))^\gamma - 1 - \gamma\pi(e^{c_1(z)} - 1)) \\ & + \int_{U_0} (1 + \pi(e^{c_2(z)} - 1))^\gamma - 1]\lambda(dz)]. \end{aligned}$$

Now assuming that $v_0 \in Q(\bar{D}_\beta)$ is a subsolution of (3.8) in $\bar{D}_\beta$ and $\bar{v} \in Q(\bar{D}_\beta)$ is a supersolution of (3.8) in D_β. Then

$$v_0 \leq \bar{v} \quad \text{on } \bar{D}_\beta.$$

Consequently,the HJB equation admits at most one constrained viscosity solution in $Q(\bar{D}_\beta)$.

3.2 Proofs

This section is solely dedicated to proof the existence of a constrained viscosity solution to (3.8). Note the regularity of v will not be considered here.

3.2.1 Proof of the existence of a viscosity solution

The proof of Th 3.1.7 will be carried out in 3 steps.

(i) *Property of v.* The value function $v(x, y)$, a non-negative function on $\bar{D}_\beta$ is well defined by the local boundedness of the utility $u(\cdot)$. As a result of proposition 1.3 in [26] we obtain the continuity of v and the local boundedness of $u(\cdot)$.

For boundedness, since we have concavity of $u(\cdot)$

$$u(y) \leq K(1+y)$$

for some large $K > 0$. Hence

$$\begin{aligned}
\mathbb{E}[\int_0^t e^{-\alpha s} u(y_s) ds] &\leq \mathbb{E}[K \int_0^t e^{-\alpha s}(1+y_s) ds] \\
&\leq K \int_0^t e^{-\alpha s}(1+M_1) ds \\
&\leq \frac{1}{\alpha}(1+M_1)K(1-e^{-\alpha t}).
\end{aligned}$$

By letting $t \to \infty$, we achieve boundedness.

(ii) *v is a Subsolution.* Let $\phi \in C^2(\bar{D}_\beta) \cap Q(\bar{D}_\beta)$ and let (x, y) be the global maximizer of $v - \phi$ in $\bar{D}_\beta$. We assume $(v - \phi)(x, y) = 0$.

We aim to prove

$$max[Nv, \sup_{\pi, g \in \mathcal{A}} Av, Mv] \geq 0,$$

so we will start by considering the contrary

$$[Nv, \sup_{\pi, g \in \mathcal{A}} Av, Mv](x, y) < 0.$$

By the continuity, there exists an open ball $B_r = B_r((x, y))$ with center (x, y) and radius $r > 0, \epsilon > 0$,and $\hat{g}$ such that on $\overline{B_r \cap D_\beta}$

$$M\phi \leq 0, \qquad N\phi \leq 0,$$

and the generator Av

$$
\begin{aligned}
&-\alpha v - \beta y\phi_y + \sigma(t)\pi x v_{xx} + max_\pi\{(r + \pi([b(t) + \frac{1}{2}\sigma(t)^2 \\
&+ \int_{U\backslash U_0} (e^{c_1(z)} - 1 - c_1(z))\lambda(dz)] - r))xv_x \\
&+ B^\pi((x,y),\phi) + u(y) - y(\phi_x - \beta\phi_y)\} \\
\le\ & -\epsilon\alpha.
\end{aligned}
$$

Then

$$
v \le \phi - \epsilon \qquad on \qquad \partial B_r \cap \bar{D}_\beta.
$$

Let $(X_0, Y_0) = (x, y)$, and let

$$
\tau^* = \inf\{t \ge 0; (X_t, Y_t) \notin B_r\}, \qquad \tau_L = \inf\{t \ge 0; \Delta_L X_t \ne 0\} > 0.
$$

Furthermore, let $\tau = min(\tau_L, \tau^*)$.

So let the optimal trajectory be denoted (X^*, Y^*) and the optimal control be denoted by (π^*, g^*, L^*). Then by considering the 2 following cases we will demonstrate the existence of the viscosity solution to (3.8);

(a) When $\{\tau^* \ge \tau_L\}, \tau = \tau_L > 0$ a.s. Then using Itô's formula for semi-martingales together with the inequalities stated above

$$
\begin{aligned}
& v(x,y) \\
=\ & \int_0^{\tau_L} e^{-\alpha s} u(y_s) ds + e^{-\alpha\tau_L} v(X_{\tau_L}, Y_{\tau_L}) \\
& \le \int_0^{\tau_L} e^{-\alpha s} u(y_s) ds + e^{-\alpha\tau_L} \phi(X_{\tau_L}, Y_{\tau_L}) \\
& \le \phi(x,y) + \int_0^{\tau_L} ds e^{-\alpha s}\{u(y_s) - \alpha e^{-\alpha s}\phi(X_s, Y_s) \\
& + (r + \pi([b(s) + \frac{1}{2}\sigma(s)^2 + \int_{U\backslash U_0} (e^{c_1(z)} - 1 - c_1(z))\lambda(dz)] - r))X_s\phi_x \\
& + \sigma(t)\pi X_s \phi_{xx} - \beta Y_s \phi_y + B^\pi((X_s, Y_s), \phi)\}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{s\in[0,\tau_L)} e^{-\alpha s}\{\phi(X_{s-}+\Delta L_s, Y_{s-})-\phi(X_{s-},Y_{s-})\}\\
&+\int_0^{\tau_L} e^{-\alpha s}\{\int_{U\backslash U_0}(\phi(X_{s-}+\pi_{s-}X_{s-}(e^{(c_1(z)}-1),Y_{s-})\\
&\qquad -\phi(X_{s-},Y_{s-}))\\
&+\int_{U_0}(\phi(X_{s-}+\pi_{s-}X_{s-}(e^{c_2(z)}-1),Y_{s-})\\
&\qquad -\phi(X_{s-},Y_{s-}))\}\tilde{N}(ds,dz),\\
&\le \phi(x,y)-\epsilon(1-e^{-\alpha\tau_L})\\
&+\int_0^{\tau_L} e^{-\alpha s}\{\int_{U\backslash U_0}(\phi(X_{s-}+\pi_{s-}X_{s-}(e^{c_1(z)}-1),Y_{s-})\\
&\qquad -\phi(X_{s-},Y_{s-}))\\
&\qquad\quad +\int_{U_0}(\phi(X_{s-}+\pi_{s-}X_{s-}(e^{c_2(z)}-1),Y_{s-})\\
&\qquad -\phi(X_{s-},Y_{s-}))\}\tilde{N}(ds,dz).
\end{aligned}
$$

(b) When $\{\tau^* < \tau_L\}$, any one of the terms G_t, L^c, or N will make the process (X_t, Y_t) move out of B_r. Let (x', y') be on the intersection of ∂B_r and the line connecting $(X_{\tau^*-}, Y_{\tau^*-})$ to (X_{τ^*}, Y_{τ^*}). The slope vector of this line is $(-1, \beta)$ or $(1, 0)$, and by Lemma 1.4 in [26] v is decreasing along this line. Therefore from the above we have, for some $\epsilon > 0$,

$$
v(X_{\tau^*}, Y_{\tau^*}) \le v(x', y') \le \phi(x', y') - \epsilon \le \phi(X_{\tau^*}, Y_{\tau^*}) - \epsilon.
$$

Then

$$
\begin{aligned}
& v(x,y) \\
= \ & \int_0^{\tau^*} e^{-\alpha s} u(y_s) ds + e^{-\alpha\tau^*} v(X_{\tau^*}, Y_{\tau^*}) \\
\leq \ & \int_0^{\tau^*} e^{-\alpha s} u(y_s) ds + e^{-\alpha\tau^*} v(X_{\tau^*}, Y_{\tau^*}) - \epsilon e^{-\alpha\tau^*} \\
& \quad + e^{-\alpha\tau^*} v(X_{\tau^*-}, Y_{\tau^*-}) - \epsilon e^{-\alpha\tau^*} \\
= \ & \int_0^{\tau^*} e^{-\alpha s} u(y_s) ds + \{\phi(x,y) \\
& + \int_0^{\tau^*} (-\alpha) e^{-\alpha s} \phi(X_s, Y_s) ds + \int_0^{\tau^*} e^{-\alpha s} \phi_x(X_s, Y_s) dX_s \\
& + \int_0^{\tau^*} e^{-\alpha s} \phi_y(X_s, Y_s) dY_s + \int_0^{\tau^*} e^{-\alpha s} \phi_{xy}(X_s, Y_s) d[X,Y]_s^y \\
& + \frac{1}{2} \int_0^{\tau^*} e^{-\alpha s} \phi_{xx}(X_s, Y_s) d[X,X]_s^y \\
& + \frac{1}{2} \int_0^{\tau^*} e^{-\alpha s} \phi_{yy}(X_s, Y_s) d[Y,Y]_s^y \\
& + \sum_{s \in [0,\tau^*)} e^{-\alpha s} \{\phi(X_s, Y_s) - \phi(X_{s-}, Y_{s-}) - (\phi_x(X_{s-}, Y_{s-}) \Delta X_s \\
& + \phi_y(X_{s-}, Y_{s-}) \Delta Y_s)\}\} - \epsilon e^{-\alpha\tau^*} \\
= \ & -\epsilon e^{-\alpha\tau^*} + \phi(x,y) \\
& + \int_0^{\tau^*} e^{-\alpha s} \{u(y_s) - \alpha e^{-\alpha s} \phi(X_s, Y_s) \\
& + (r + \pi([b(s) + \frac{1}{2}\sigma(s)^2 + \int_{U \setminus U_0} (e^{c_1(z)} - 1 - c_1(z)) \lambda(dz)] - r)) X_s \phi_x \\
& + \sigma(t) \pi X_s \phi_{xx} - \beta Y_s \phi_y + B^\pi((X_s, Y_s), \phi)\} ds \\
& + \int_0^{\tau^*} e^{-\alpha s} (-\phi_x + \beta \phi_y)(X_s, Y_s) Y_s ds + \int_0^{\tau^*} e^{-\alpha s} \phi_x(X_s, Y_s) dL_t^c \\
& + \sum_{s \in [0,\tau^*)} e^{-\alpha s} \{\phi(X_{s-} + \Delta L_s, Y_{s-} - \gamma \Delta L_s) - \phi(X_{s-}, Y_{s-})\} \\
& + \int_0^{\tau^*} e^{-\alpha s} \{ \int_{U \setminus U_0} (\phi(X_{s-} + \pi_{s-} X_{s-} (e^{c_1(z)} - 1), Y_{s-}) \\
& \qquad\qquad - \phi(X_{s-}, Y_{s-})) \\
& + \int_{U_0} (\phi(X_{s-} + \pi_{s-} X_{s-} (e^{c_2(z)} - 1), Y_{s-}) \\
& \qquad\qquad - \phi(X_{s-}, Y_{s-}))\} \tilde{N}(ds, dz).
\end{aligned}
$$

On the other hand, since $M\phi \leq 0$ implies $-\phi_x + \beta\phi_y \leq 0$ on $\{x \geq 0\}$, and since $N\phi \leq 0$ implies $\phi_x \leq 0$ on $\{x \leq 0\}$, $\phi(X_{s-} + \Delta L_s, Y_{s-}) - \phi(X_{s-}, Y_{s-}) \leq 0$, and since $-\alpha\phi \leq -\alpha v - \epsilon\alpha \leq -\alpha v$, we have

$$
\begin{aligned}
& R.H.S. \\
\leq\ & \phi(x,y) - \epsilon e^{-\alpha\tau^*} + \int_0^{\tau^*} e^{-\alpha s}(-\epsilon\alpha)ds \\
& + \int_0^{\tau^*} e^{-\alpha s}\{\int_{U\setminus U_0} e^{-\alpha s}(\phi(X_{s-} + \pi_{s-}X_{s-}(e^{c_1(z)} - 1), Y_{s-}) \\
& \qquad -\phi(X_{s-}, Y_{s-})) \\
& + \int_{U_0} (\phi(X_{s-} + \pi_{s-}X_{s-}(e^{c_2(z)} - 1), Y_{s-}) \\
& \qquad -\phi(X_{s-}, Y_{s-}))\}\tilde{N}(ds, dz). \\
\leq\ & \phi(x,y) - \epsilon + \int_0^{\tau^*} e^{-\alpha s}\{\int_{U\setminus U_0} e^{-\alpha s}(\phi(X_{s-} + \pi_{s-}X_{s-}(e^{c_1(z)} - 1), Y_{s-}) \\
& \qquad -\phi(X_{s-}, Y_{s-})) \\
& + \int_{U_0} (\phi(X_{s-} + \pi_{s-}X_{s-}(e^{c_2(z)} - 1), Y_{s-}) \\
& \qquad -\phi(X_{s-}, Y_{s-}))\}\tilde{N}(ds, dz).
\end{aligned}
$$

Now from the previous two cases (a) and (b) , we obtain

$$
\begin{aligned}
& \mathbb{E}[\int_0^{\tau} e^{-\alpha s}u(y_s)ds + e^{-\alpha\tau}v(X_\tau, Y_\tau)] \\
\leq\ & \mathbb{E}[1_{\{\tau^* < \tau_L\}} \cdot (\int_0^{\tau *} e^{-\alpha s}u(y_s)ds + e^{-\alpha\tau^*}v(X_{\tau^*}, Y_{\tau^*}))] \\
& + \mathbb{E}[1_{\{\tau^* \geq \tau_L\}} \cdot (\int_0^{\tau_L} e^{-\alpha s}u(y_s)ds + e^{-\alpha\tau_L}v(X_{\tau_L}, Y_{\tau_L}))] \\
\leq\ & \phi(x,y) - \epsilon\mathbb{E}[1 - 1_{\{\tau^* \geq \tau_L\}} \cdot e^{-\alpha\tau_L}] \\
\leq\ & \phi(x,y) - \epsilon\mathbb{E}[1 - e^{-\alpha\tau_L}].
\end{aligned}
$$

Now by the Bellman principle outlined in [26],

$$v(x,y) = \sup_{(\pi,g,L)\in\mathcal{A}} \mathbb{E}[\int_0^{\tau\wedge t} e^{-\alpha s}u(y_s)ds + e^{-\alpha(\tau\wedge t)}v(X_{\tau\wedge t}, Y_{\tau\wedge t})] \tag{3.10}$$

where $v(x,y) = \phi(x,y)$, thus we have as a result of Lemma 2.1 in [26] a contradiction by letting $t \to \infty$.

(iii) *v is a supersolution.* Let $\phi \in C^2(\bar{D}_\beta) \cap Q(\bar{D}_\beta)$, and let $(x,y) \in D_\beta$ be the global minimizer of $v - \phi$ in $\bar{D}_\beta$. We assume $(v-\phi)(x,y) = 0$. Then by Lemma 1.4 in [26]

$$\phi(x,y) = v(x,y) \geq v(x-m+l, y+\beta m) \geq \phi(x-m+l, y+\beta m).$$

Hence

$$0 \geq \phi((x,y) + m(-1,\beta) + l(1,0)) - \phi(x,y).$$

Dividing by m(resp. l) and letting $m \to 0$(resp. $l \to 0$)), we get

$$-\phi_x + \beta\phi_y \leq 0(Mv \leq 0) \qquad \phi_x \leq 0(Nv \leq 0). \tag{3.11}$$

Let τ_r be the exit time from $B_r = B_r((x,y))$. If we apply the Bellman principle (Lemma 1.5 from [26]) with $\pi_t = \pi, g_t = 0, \tau = \tau_r \wedge h$. Furthermore, by the assumption $v(x,y) = \phi(x,y)$, we obtain

$$\begin{aligned}
0 \quad &\geq \mathbb{E}[\int_0^{\tau\wedge t} e^{-\alpha s}u(y_s)ds + e^{-\alpha(\tau\wedge t)}\phi(X_{\tau\wedge t}, Y_{\tau\wedge t})] - \phi(x,y) \\
\geq \quad &\mathbb{E}[\int_0^{\tau\wedge t} e^{-\alpha s}\{u(y_s) - \alpha\phi - \beta Y_s\phi_y \\
&+(r + ([b(s) + \frac{1}{2}\sigma(s)^2 + \int_{U\setminus U_0}(e^{c_1(z)} - 1 - c_1(z))\lambda(dz)] - r)\pi)X_s\phi_x \\
&+\sigma(s)\pi X_s\phi_{xx} + B^\pi((x,y),\phi)\}ds]
\end{aligned}$$

$$
\begin{aligned}
\geq\ & \mathbb{E}[(\frac{1}{\alpha})(1-e^{-\alpha(h\wedge\tau_r)})].\inf_{(x,y)\in B_r}[u(y)-\alpha\phi-\beta y\phi_y \\
& +(r+([b(t)+\frac{1}{2}\sigma(t)^2+\int_{U\backslash U_0}(e^{c_1(z)}-1-c_1(z))\lambda(dz)]-r)\pi)x\phi_x \\
& +\sigma(t)\pi x\phi_{xx}+B^{\pi}((x,y),\phi)].
\end{aligned}
$$

By the right continuity of the paths , $\tau_r > 0$ a.s. Hence

$$
lim_{h\to 0}\mathbb{E}[(\frac{1}{h})(1-e^{-\alpha(h\wedge\tau_r)})] = \alpha.
$$

If we divide the above inequality by h, then allowing $h \to 0$ and $r \to 0$, we obtain

$$
\begin{aligned}
& u(y)-\alpha\phi-\beta y\phi_y+\sigma(t)\pi x\phi_{xx}+(r+([b(t)+\frac{1}{2}\sigma(t)^2 \\
& +\int_{U\backslash U_0}(e^{c_1(z)}-1-c_1(z))\lambda(dz)])-r)\pi)x\phi_x \\
& +B^{\pi}((x,y),\phi) \\
\leq\ & 0,
\end{aligned} \tag{3.12}
$$

for every $\pi \in [0,1]$. Now as a result of (3.11), this implies that v is a viscosity supersolution. Thus we have proved the existence of a viscosity solution to (3.8).

$$QED$$

Remark 3.2.1 *A proof of the uniqueness theorem is highlighted in [26], however they consider Partial differential equations(PDEs) with first order derivatives. In order to prove the uniqueness for our situation, when considering PDEs with second order derivatives it would be required to consider the "maximum principle for semicontinuous functions", outlined in [16]. While the integro-PDE analog of this theorem is developed in [30, 31].*

Chapter 4

Explicit construction of SDEs associated with polar-decomposed Lévy measures and application to stochastic optimization.

4.1 The construction of the coefficient of the jump term

It was demonstrated in section 2.1 that it is possible to construct a jump-type stochastic differential equation(SDE) starting with a Lévy generator having variable coefficients in its full general form

$$
\begin{aligned}
Lf(t,x) &= \frac{1}{2}\sum_{i,j=1}^{d} a^{i,j}(t,x)\partial_i\partial_j f(t,x) + \sum_{i=1}^{d} b^i(t,x)\partial_i f(t,x) \qquad (4.1)\\
&+ \int_{\mathbb{R}^d\setminus\{0\}} \{f(t,x+z) - f(t,x) - \frac{z1_{\{|z|<1\}}\cdot\nabla f(t,x)}{1+|z|^2}\}\nu(t,x,dz),
\end{aligned}
$$

where $a(t,x) = (a^{i,j}(t,x))$ is a non-negative definite symmetric $d\times d$-matrix-valued measurable function on $[0,\infty)\times\mathbb{R}^d, b(t,x) = (b^i(t,x))$ is a $\mathbb{R}^d$-valued measurable function on $[0,\infty)\times\mathbb{R}^d$ and $\nu(t,x,dz)$ is a Lévy kernel on $[0,\infty)\times\mathbb{R}^d\times\mathcal{B}(\mathbb{R}^d\setminus\{0\})$. Let us give a brief account on this point.

We start with a probability set-up $(\Omega,\mathcal{F},P;\{\mathcal{F}_t\}_{t\in[0,\infty)})$. Given a σ-finite measure space $(U,\mathcal{B}(U),\lambda)$, one can construct a canonical Poisson random measure N (cf. e.g. [25])

$$
N : \mathcal{B}([0,\infty))\times\mathcal{B}(U)\times\Omega \to \mathbb{N}\cup\{0\}\cup\{\infty\},
$$

with intensity measure λ. Moreover, one can have a bimeasurable bijection $c : [0,\infty)\times\mathbb{R}^d\times U \to \mathbb{R}^d\setminus\{0\}$ such that (cf. [17])

$$
\int_U 1_A(c(t,x,y))\lambda(dy) = \int_{\mathbb{R}^d\setminus\{0\}} 1_A(z)\nu(t,x,dz), \tag{4.2}
$$

for any $A\in\mathcal{B}(\mathbb{R}^d\setminus\{0\})$ and $\forall(t,x)\in[0,\infty)\times\mathbb{R}^d$. By utilizing this relation, one can construct a jump-type SDE (cf. e.g. [39, 48, 10]) associated with L given in (4.1) as follows

$$
dS_t = b(t,S_t)dt + \sigma(t,S_t)dW_t + \int_U c(t,S_{t-},y)\tilde{N}(dt,dy), \tag{4.3}
$$

where $b : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ and $c : [0,\infty)\times\mathbb{R}^d\times U \to \mathbb{R}^d$ are measurable as given respectively in (4.1) and (4.2), W_t is an m-dimensional $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-Brownian motion with $m\in\mathbb{N}$ being fixed, $\sigma : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^{d\otimes m}$ is measurable such that $\sigma(t,x)\sigma^T(t,x) = a(t,x)$, and $\tilde{N}$ is the compensating

$\{\mathcal{F}_t\}_{t\in[0,\infty)}$-martingale measure associated of the canonical Poisson random measure N, namely,

$$\tilde{N}(dt, dy, \omega) := N(dt, dy, \omega) - dt\lambda(dy).$$

In this construction it should be noted that the diffusion matrix a and the Lévy kernel ν do not preserve the original form, so as a result we aim to seek the conditions imposed, and under which the existence and uniqueness of a solution to Equation (4.3) hold. A well known sufficient condition for the existence and uniqueness of solutions is that the coefficients satisfy the Lipschitz condition and the linear growth condition (cf e.g. Theorem 1.1 in [39]).

1. (*The linear growth condition*) There exists $C > 0$ such that for all $x \in \mathbb{R}^d$,

$$|b(t,x)|^2 + |\sigma(t,x)|^2 + \int_U |c(t,x,y)|^2\lambda(dy) \leq C(1+|x|^2). \tag{4.4}$$

2. (*The Lipschitz condition*) There exists $C > 0$ such that for all $x, x' \in \mathbb{R}^d$,

$$\begin{aligned}|b(t,x) - b(t,x')|^2 + |\sigma(t,x) - \sigma(t,x')|^2 \\ + \int_U |c(t,x,y) - c(t,x',y)|^2\lambda(dy) \leq C|x-x'|^2.\end{aligned} \tag{4.5}$$

A general condition imposed on the diffusion matrix for the existence of the associated SDE with a Lipschitz continuous coefficient was obtained by Phillips and Sarason [42] and Freidlin [20], one can also refer to [25], however, our focus is primarily on the jump coefficient. The coefficient of the jump term can be obtained from the Lévy measure itself (c.f. [27, 22, 17]). However, in the general case it remains unknown whether or not the coefficient constructed ensures uniqueness of solutions, this is due to the Lipschitz continuity of the coefficient not being achieved.

The intention here is to construct the coefficient of the jump term and to oultine the conditions such that a unique solution to Equation (4.3) does exist. To this end, from now on we shall consider those Lévy measures ν with polar decomposition. That is, we shall work with those Lévy measure ν having the representation (4.6) below. We then move to considering the particular situation that the given Lévy measure is that of a stable-like process. As a result we will be able to define explicitly the coefficient of the jump term, which will then allow us to apply our explicit construction to two portfolio optimization problems over $\mathbb{R}^d$, for the cases when $d = 1$ and when $d \geq 2$, respectively.

The construction of the coefficient of the jump term will be done by virtue of polar decomposition of the Lévy measure ν. In section 4.2 it will be demonstrated how the polar decomposition of the given Lévy measure relates to that of a stable-like process, for the cases when $d = 1$ and $d \geq 2$, respectively.

We work with the following setting. Let $S(d\theta)$ be a finite Borel measure on the unit sphere S^{d-1}. Suppose we are given an $\mathbb{R}^d$-valued, measurable function $z : \mathbb{R}^d \times S^{d-1} \times [0, \infty) \to \mathbb{R}^d$ such that $z(\cdot, \cdot, 0) = 0$ and $\forall x \in \mathbb{R}^d, z(x, \cdot, \cdot)$ is a bimeasurable bijection from $S^{d-1} \times (0, \infty) \to \mathbb{R}^d \setminus \{0\}$, and a positive kernel $g(x, \theta, d\rho)$, where θ and ρ stand for the polar coordinates of $\mathbb{R}^d$, then a Lévy measure ν has a polar decomposition if the following holds

$$\nu(x, A) = \int_{S^{d-1}} S(d\theta) \int_{(0,\infty)} 1_A(z(x, \theta, \rho)) g(x, \theta, d\rho), \quad x \in \mathbb{R}^d. \tag{4.6}$$

As was pointed out earlier, the coefficient of the jump term in Equation (4.3) is obtained from the Lévy measure ν, so if we let

$$\hat{c}(x,\theta,r) := z(x,\theta,G^{-1}(x,\theta,r)), \tag{4.7}$$

and choose g and G to be bounded functions which have the relation

$$G(x,\theta,\eta) := \int_{(\eta,\infty)} g(x,\theta,d\rho), \quad \eta \in (0,\infty),$$

and let $G^{-1}(x,\theta,\cdot)$ denote the right continuous inverse function of $G(x,\theta,\cdot)$ such that

$$G^{-1}(x,\theta,r) = \inf\{\eta \in (0,\infty) : G(x,\theta,\eta) \leq r\}.$$

Thus by realising our previous σ-finite measure space $(U,\mathcal{B}(U),\lambda)$ as

$$(U,\mathcal{B}(U),\lambda) := (S^{d-1}\times(0,\infty),\mathcal{B}(S^{d-1}\times(0,\infty)),\lambda),$$

and by virtue of Relation (4.2) for the determination of the measure λ on the measurable space $(S^{d-1}\times(0,\infty),\mathcal{B}(S^{d-1}\times(0,\infty))$, we see that $\hat{c}$ as defined by (4.7) is the coefficient of the jump term in the jump SDE (4.3) associated with the polar-decomposed Lévy measure $\nu(x,dz)$.

Naturally, we now move to outlining the conditions set upon the given curves z and the weights g such that there exists a coefficient that satisfies the Lipschitz condition and the linear growth condition(cf Theorem 1.1 in [39]).

For simplicity, we will assume $g(x,\theta,(l,+\infty)) = 0$ for some $l > 0$. Therefore ν having the following decomposition

$$\nu(x,A) = \int_{S^{d-1}} S(d\theta)\int_{(0,l]} 1_A(z(x,\theta,\rho))g(x,\theta,d\rho),$$

for $A \in \mathcal{B}(S^{d-1}\times(0,\infty))$.

Let $G:\mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz continuous function and let the upper gradient of G be denoted by $\nabla_x G$:

$$\nabla_x G(x) := (D_1G(x),......,D_dG(x)),$$

where

$$D_jG(x) := \limsup_{h\to 0} |\frac{1}{h}\{G(x+he_j) - G(x)\}|,$$

with e_j being the unit vector in the j-direction. Let us present our assumptions on z and g as follows:

Conditions on $z(x,\theta,\rho)$. The measurable function $z : \mathbb{R}^d \times S^{d-1} \times [0,\infty) \to \mathbb{R}^d$ satisfies:

1. There exists $C_1 > 0$ such that

$$\sup_{x\in\mathbb{R}^d, \theta\in S^{d-1}} |z(x,\theta,\rho)| \le C_1\rho,$$

   for $\rho \in [0,l]$.

2. For every $R > 0$ there exists C_R such that

$$\sup_{\theta\in S^{d-1}} |z(x,\theta,\rho) - z(x',\theta,\rho)| \le C_R|x - x'|\rho,$$

   for $|x| \le R, |x'| \le R$ and $\rho \in [0,l]$.

3. For every $R > 0$ there exists a positive constant C_R such that

$$\sup_{|x|\le R, \theta\in S^{d-1}} |z(x,\theta,\rho) - z(x,\theta,\rho')| \le C_R|\rho - \rho'|,$$

   for $\rho, \rho' \in [0,l]$.

Conditions on $g(x,\theta,d\rho)$. The positive kernel g satisfies:

1. There exists $B_1 > 0$ such that for $x \in \mathbb{R}^d$

$$\int_{S^{d-1}} S(d\theta) \int_{(0,l]} \rho^2 g(x,\theta,d\rho) \le B_1(1 + |x|^2).$$

2. $G(x, \theta, \eta)$ is finite for all $x \in \mathbb{R}^d, \theta \in S^{d-1}$ and $\eta \in (0, l]$, and also for any $x \in \mathbb{R}^d$ and $\theta \in S^{d-1}$,

$$G(x, \theta, 0+) = +\infty \quad and \quad G(x, \theta, l-) = 0.$$

Moreover, for each θ and $\eta, G(x, \theta, \eta)$ is locally Lipschitz continuous in x: for any $R > 0, \eta_0 \in (0, l)$ it satisfies

$$\sup_{\theta \in S^{d-1}, |x|, |x'| \leq R, \eta_0 \leq \eta \leq l} \frac{|G(x, \theta, \eta) - G(x', \theta, \eta)|}{|x - x'|} < +\infty.$$

3. Let $g_{ac}(x, \theta, \rho) d\rho$ be the absolute continuous part of $g(x, \theta, d\rho)$. Let there exist a positive function $g_0(x, 0, \rho)$ satisfying

$$g_0(x, 0, \rho) \leq K_R g_{ac}(x, 0, \rho),$$

for every $|x| \leq R, \theta \in S^{d-1}$ and $\rho \in (0, l]$, and note $g_0(x, \theta, \rho)$ is continuous in $(x, \rho) \in \mathbb{R}^d \times (0, l]$ for each $\theta \in S^{d-1}$.

4. For every $R > 0$,

$$\sup_{|x| \leq R} \int_{S^{d-1}} S(d\theta) \int_{(0,l]} \frac{| \nabla_x G((x, \theta, \eta)|^2}{g_0(x, \theta, \eta)^2} g(x, \theta, d\eta) < +\infty.$$

Now let us summarize our discussion as the following proposition.

Proposition 4.1.1 *The coefficient of the jump term $\hat{c}$ defined by (4.7) satisfies the linear growth condition and the local Lipschitz condition with respect to the measure $\lambda(dy)$ provided the conditions imposed on the curves z and the weight g are satisfied.*

The proof of this proposition is similar to that outlined in [39]. As a result of this proposition and the notions of theorem 1.1 in [39], there exists a unique solution to the associated SDE (4.3).

Having constructed $\hat{c}(x, \theta, r)$ and stated all the conditions that we impose on its parameters, we now can move to demonstrating how the polar-decomposed Lévy measure $\nu(x, dz)$ relates to the Lévy measure of a stable-like process.

4.2 The Results

In this section we will construct the coefficient of the jump term $\hat{c} : \mathbb{R}^d \times S^{d-1} \times (0, \infty) \to \mathbb{R}^d$ explicitly when dealing with a given Lévy measure which has a stable-like representation. It will also be demonstrated how this Lévy measure relates to the polar-decomposed Lévy measure (4.6).

We start with a given Lévy measure ν on $\mathbb{R}^d$ that satisfies

$$\int_{\mathbb{R}^d \setminus \{0\}} \frac{|z|^2}{1 + |z|^2} \nu(x, dz) < \infty, \tag{4.8}$$

and has the polar decomposition

$$\nu(x, dz) = g(x, \theta, d\rho) S(d\theta) \tag{4.9}$$

where $S(d\theta)$ is a finite Borel measure on S^{d-1} and $g(x, \theta, d\rho)$ is the positive kernel which satisfies the four conditions outlined in section 4.1. Moreover, for each $(x, \theta) \in \mathbb{R}^d \times S^{d-1}$ a finite Borel measure on $(0, \infty)$ satisfying

$$\int_0^\infty \frac{\rho^2}{1 + \rho^2} g(x, \theta, d\rho) < \infty.$$

Note that Equality (4.9) clearly conforms to (4.6) with the characteristic function $1_{dz}(z(x, \theta, \rho))$ being bounded by 1.

Furthermore, for the given Lévy measure $\nu(x, dz)$ to be that of a stable-like process with index $\alpha(x)$ it must satisfy condition (4.8) and have the following polar decomposition

$$\nu(x,dz) = \frac{d\rho}{\rho^{1+\alpha(x)}} S(d\theta), \quad \alpha(x) \in (0,2), x \in \mathbb{R}^d, \rho \in (0,\infty). \tag{4.10}$$

To this end, make the positive kernel $g(x,\theta,d\rho)$ to be the following

$$g(x,\theta,d\rho) = \frac{d\rho}{\rho^{1+\alpha(x)}}. \tag{4.11}$$

This shows how the polar decomposition of the given Lévy measure ν (4.6) relates to the concrete case when given a Lévy measure of a stable-like process (4.10).

With a particular interest in using stable-like processes to model the price of an asset in the financial market, we require some conditions, therefore we will assume the conditions outlined in section 1.3.2. Since we are dealing with an index that is dependent on its starting point $x \in \mathbb{R}^d$, we need this regularity condition which states that $\alpha(x)$ needs to be sufficiently far away from 0 and 2.

$$0 < \inf_{x \in \mathbb{R}^d} \alpha(x) \leq \sup_{x \in \mathbb{R}^d} \alpha(x) < 2. \tag{4.12}$$

With this condition we avoid any singularities which can be created. Along with this condition we have an additional condition (4.13) which is bounded from below, this makes sure that $\alpha(x) \neq 2$, because when $\alpha(x) = 2$, we get degenerate Gaussianity which creates unwanted singularities.

$$2 - \alpha(x) \geq A(1 + |x|^2)^{-1}. \tag{4.13}$$

We now move to the explicit construction of the coefficient of the jump term for the cases when $d \geq 2$ and when $d = 1$. In order to do this we continue in the setup considered in section 4.1. We will highlight our results by the following two theorems which will aptly be utilised to solve the two portfolio optimization problems concretely.

Since any point in the Euclidean space $\mathbb{R}^d$ has a polar representation, we are able to employ our construction of the coefficient of the jump term in order to solve these portfolio optimization problems.

Theorem 4.2.1 *For $d \geq 2$, i.e., for the case that the given σ-finite measure space*

$$(U, \mathcal{B}(U), \lambda) = (S^{d-1} \times (0, \infty), \mathcal{B}(S^{d-1} \times (0, \infty)), \lambda)$$

the coefficient of the jump term in the SDE associated to $\nu(x, dz)$ defined by (4.10) is given by $\hat{c} = \eta\theta$.

Proof. We start with a Lévy measure ν which has polar decomposition with the curves $z(x, \theta, \rho)$ and weights $g(x, \theta, d\rho)$ as in (4.6) such that given

$$z(x, \theta, \rho) = \rho\theta, \quad \rho \in (0, \infty), \theta \in S^{d-1}$$

where

$$\rho = |z(x, \theta, \rho)|, \qquad \theta = \frac{z(x, \theta, \rho)}{|z(x, \theta, \rho)|}.$$

Let the positive kernel $g(x, \theta, d\rho)$ be defined by (4.11) then $\nu(x, dz)$ is the Lévy measure of a stable-like process if the following polar decomposition holds

$$\nu(x, dz) = \frac{d\rho}{\rho^{1+\alpha(x)}} S(d\theta), \quad x \in \mathbb{R}^d. \tag{4.14}$$

Thus, by (4.7) the coefficient of the jump term $\hat{c}$ is defined

$$\hat{c}(x, \theta, r) := z(x, \theta, G^{-1}(x, \theta, r)).$$

From *Conditions on g* (2) and (3) it can be seen that $G(x, \theta, \cdot)$ is strictly decreasing and $G(x, \theta, 0+) = +\infty$, as a result $G^{-1}(x, \theta, \cdot)$ is positive, continuous and decreasing on $[0, \infty)$. Therefore, for each x and θ

$$G^{-1}(x, \theta, G(x, \theta, \eta)) = \eta, \quad \forall \eta \in (0, \infty).$$

Therfore

$$\begin{aligned} \hat{c}(x, \theta, \eta) &:= z(x, \theta, G^{-1}(x, \theta, \eta)) \\ &= z(x, \theta, G^{-1}(x, \theta, G(x, \theta, \eta)) \\ &= z(x, \theta, \eta) \\ &= \eta\theta. \end{aligned} \tag{4.15}$$

QED

For the case when $d = 1$, we have the following theorem;

Theorem 4.2.2 *For the case that the given σ-finite measure space*

$$(U, \mathcal{B}(U), \lambda) = ((0, \infty), \mathcal{B}((0, \infty)), \lambda)$$

the coefficient of the jump term in the SDE associated to $\nu(x, dz)$ defined by

$$\nu(x, dz) := \frac{d\rho}{\rho^{1+\alpha(x)}} \quad \alpha(x) \in (0, 2), \quad x \in \mathbb{R}$$

is given by $\hat{c} = \eta$.

Proof. Let us start with a Lévy measure ν which has polar decomposition with the curves $z(x, \rho)$ and weights $g(x, d\rho)$ as in (4.6) such that given

$$z(x, \rho) = \rho, \quad \rho \in (0, \infty)$$

and the positive kernel $g(x, d\rho)$ defined by (4.11) (note there is no angular dependence when d=1). Therefore $\nu(x, dz)$ is the Lévy measure of a stable-like process if it has the following polar decomposition

$$\nu(x, dz) = g(x, d\rho) = \frac{d\rho}{\rho^{1+\alpha(x)}} \quad \alpha(x) \in (0, 2), \quad x \in \mathbb{R}. \tag{4.16}$$

Hence by a similar arguement as in the proof of Theorem 4.2.1. By *Conditions on g* (2) and (3) it can be seen that $G(x, \cdot)$ is strictly decreasing and $G(x, 0+) = +\infty$, as a result $G^{-1}(x, \cdot)$ is positive, continuous and decreasing on $[0, \infty)$. Therefore, for each x

$$\begin{aligned} \hat{c}(x, \eta) &:= z(x, G^{-1}(x, \eta)) \\ &= z(x, G^{-1}(x, G(x, \eta))) \\ &= z(x, \eta) \\ &= \eta, \qquad \eta \in (0, \infty). \end{aligned} \tag{4.17}$$

QED

4.3 Application to financial optimization problems

The intention for this final section is to solve concretely the two financial optimization problems previously considered in an abstract setting, where the financial market model is now being driven by a stable-like process. With stable-like processes being the driving force we will assume throughout all the condtions outlined in section 1.3.2 and those outlined in this chapter.

The consideration throughout this research has been in a general setting, and now that we have constructed the coefficient of the jump term explicitly when $\nu(x, dz)$ is a Lévy measure of a stable-like process we can with the aid of the bridge equality (4.2) realize these optimization problems concretely.

We will use the sufficient maximum principle and the results from section 4.2 to solve the consumption-portfolio optimization problem consider in chapter 2. In chapter 3, recall we considered a portfolio optimization problem for a pair consisting of the wealth process and the cumulative comsumption process in an incomplete financial market model. Notably in these 2 problems the financial market will be driven by stable-like processes due to our concrete Lévy measures.

The most important tool which allows us to consider bridging from a general σ-finite measure space $(U, \mathcal{B}(U), \lambda)$ into the measure space $(\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\}, \nu)$ is the bridge equality. Using the ideas presented by El-Karoui and Lepeltier [17], which state that the function $c : \mathbb{R}^d \times U \to \mathbb{R}^d$ is a $one - to - one$ and *Onto* function, thus implying the existence of its inverse function. Now this inverse function is indeed the coefficient of the jump term $\hat{c}(x, \theta, r)$ we constructed in section 4.1 and explicitly in section 4.2 (cf Theorems 4.2.1, 4.2.2).

4.3.1 Consumption-portfolio optimization problem

To solve such an optimization problem explicitly when $\nu(x, dz)$ is stable-like, we will refer to the case when $d \geq 2$ and Theorem 4.2.1. Since we established from the ideas presented in [17] that the function c has an inverse which is the coefficient of the jump term $\hat{c}$, the bridge equality (4.2) can be modified to make the relation

$$
\begin{aligned}
\int_U 1_A(c(x,y))\lambda(dy) &= \int_{\mathbb{R}^d\setminus\{0\}} 1_A(z)\nu(x,dz) \\
&= \int_{S^{d-1}\times(0,\infty)} 1_A(\rho\theta)\frac{d\rho}{\rho^{1+\alpha(x)}}S(d\theta),
\end{aligned}
$$

where $A \in \mathcal{B}(S^{d-1} \times (0,\infty))$.

Therefore the coefficient of the jump term is the following

$$\hat{c}(\theta, r) = r\theta, \quad r \in (0,\infty), \theta \in S^{d-1}.$$

We now move to considering the Consumption-portfolio optimization with a terminal condition problem, concretely!

If we allow the agent to withdraw consumption from their wealth, this being modelled by the SDE

$$
\begin{aligned}
dR(t) &= \{\rho_t R(t) + (b(t) - \rho_t)u(t) - w(t)\}dt + \sigma(t)u(t)dW(t) \quad (4.18) \\
&\quad + u(t-)\int_{0<|r|<1} r_1\theta\tilde{N}(dt,dz) + u(t-)\int_{|r|\geq 1} r_2\theta N(dt,dz).
\end{aligned}
$$

Our objective is to solve the following consumption-portfolio optimization problem:

$$\sup_{(w,u)\in\mathcal{A}} \mathbb{E}[\int_0^T \exp(-\int_0^t \delta(s)ds)[\frac{w(t)^\gamma}{\gamma}]dt], \quad (4.19)$$

where $\mathcal{A}$ is the set of predictable consumption-portfolio pairs $d = (w,u)$ with u being tame and w being nonnegative, such that (4.18) has a strong solution over $[0,T]$. In the expression (4.19), $\delta : [0,T] \to \mathbb{R}$ is a given bounded deterministic function and $\gamma < 1$ is a given nonzero constant.

The sufficient maximum principle outlined in Theorem 2.3.1 will be used to solve this control problem with the terminal condition

$$\mathbb{E}[(\hat{R}(T) - R(T))\hat{p}(T)] \leq 0. \quad (4.20)$$

By considering $R - \bar{x}$ instead of R, where $\bar{x}$ is the nonzero minimal terminal wealth coefficient, the Hamiltonian we considered is of the following form

$$\begin{aligned}
& A(t, x, w, u, p, q, n^{(1)}, n^{(2)}) \\
= \; & \exp(-\int_0^t \delta(s)ds)(\frac{w^\gamma}{\gamma}) - pw + \rho px \\
& + u\{p(b-\rho) + q\sigma + \int_{0<|r|<1} n^{(1)}(t-, z)r_1\theta \qquad (4.21) \\
& + \int_{|r|\geq 1} n^{(2)}(t-, z)r_2\theta + r_2\theta.p\}\nu(x, dz) .
\end{aligned}$$

On the other hand, the modified adjoint equation now becomes

$$\begin{aligned}
dp(t) \;\; = \;\; & -\rho(t)p(t)dt + q(t)dW(t) + \int_{0<|r|<1} n^{(1)}(t-, z)d\tilde{N}(dt, dz) \\
& + \int_{|r|\geq 1} n^{(2)}(t-, z)dN(dt, dz). \qquad (4.22)
\end{aligned}$$

Now let the pair $(\hat{w}, \hat{u}) \in \mathcal{A}$ have corresponding solution $\hat{R}$ and $(\hat{p}, \hat{q}, \hat{n}^{(1)}, \hat{n}^{(2)})$ of equations (4.18) and (4.22).

The value of w which maximizes $A(t, \hat{R}(x), \hat{u}, \hat{p}(t), \hat{q}(t), \hat{n}^{(1)}(t, \cdot), \hat{n}^{(2)}(t, \cdot))$ is given by

$$w = \hat{w}(t) = [\exp(\int_0^t \delta(s)ds)\hat{p}(t)]^{\frac{1}{(\gamma-1)}}. \qquad (4.23)$$

Since the Hamiltonian A contains u in a linear form, it is natural to assume that the coefficient of u will vanish due to the maximum principle. Hence we have the following

$$\begin{aligned}
\hat{p}(t)(b(t) \;\; - \;\; & \rho_t) + \sigma(t)\hat{q}(t) + \int_{0<|r|<1} r_1\theta\hat{n}^{(1)}(t, z)\nu(x, dz) \\
& + \int_{|r|\geq 1} \{r_2\theta\hat{n}^{(2)}(t, z) + r_2\theta.\hat{p}(t)\}\nu(x, dz) = 0. \qquad (4.24)
\end{aligned}$$

It is also assumed that it is optimal to consume at a rate proportional to the current wealth $\hat{R}(t)$. If we set

$$\hat{p}(t) = f(t)\hat{R}(t)^{(\gamma-1)}, \tag{4.25}$$

for a deterministic differentiable function f. Now differentiating (4.25) and applying Itô's formula, we can obtain expressions for the other 3 adapted processes

$$\hat{n}^{(1)}(t,z) = f(t)\hat{R}(t)^{\gamma-1}\{(1+r_1\theta\hat{u}(t)\hat{R}(t)^{-1})^{\gamma-1} - 1\}, \tag{4.26}$$

$$\hat{n}^{(2)}(t,z) = f(t)\hat{R}(t)^{\gamma-1}\{(1+r_2\theta\hat{u}(t)\hat{R}(t)^{-1})^{\gamma-1} - 1\}, \tag{4.27}$$

$$\hat{q}(t) = (\gamma-1)f(t)\sigma(t)\hat{u}(t)\hat{R}(t)^{\gamma-2}, \tag{4.28}$$

and

$$f'(t) + \alpha_t f(t) + (1-\gamma)\exp\{\int_0^t [\frac{\delta(s)}{(\gamma-1)}]ds\} f(t)^{\frac{\gamma}{(\gamma-1)}} = 0, \tag{4.29}$$

where α_t is defined

$$\begin{aligned}\alpha_t \; = \; & \gamma\rho_t + (\gamma-1)(b(t)-\rho_t)\hat{u}(t)\hat{R}(t)^{-1} \\ & + \frac{1}{2}(\gamma-1)(\gamma-2)\sigma^2(t)\hat{u}(t)^2\hat{R}(t)^{-2} \\ & + \int_{0<|r|<1} \{(1+r_1\theta\hat{u}(t)^{-1})^{\gamma-1} \\ & \qquad -1-(\gamma-1)\hat{u}(t)\hat{R}(t)^{-1}r_1\theta\}\nu(x,dz).\end{aligned} \tag{4.30}$$

The next step is to substitute (4.26), (4.27) and (4.28) into (4.24), but before we do that we are going to allow a drift transformation

$$\tilde{b(t)} := b(t) + \int_{|r|\geq 1} r_2\theta\nu(x,dz).$$

If we set

$$F(\hat{u}(t)\hat{R}(t)^{-1}) = 0,$$

where

$$
\begin{aligned}
F(\pi) \; &:= \; \tilde{b}(t) - \rho_t + \sigma^2(t)(\gamma-1)\pi \\
&+\{\int_{0<|r|<1} r_1\theta\{(1+r_1\theta.\pi)^{\gamma-1}-1\} \\
&+\int_{|r|\geq 1} r_2\theta\{(1+r_2\theta.\pi)^{\gamma-1}-1\}\}\nu(x,dz).
\end{aligned}
$$

On the other hand, note that $F(0) = \tilde{b_t} - \rho_t > 0$. Therefore, there exists $\hat{\pi}(t) > 0$ such that

$$
F(\hat{\pi}(t)) = 0, \tag{4.31}
$$

since $F(\pi) < 0$ for large π. Let us take

$$
\hat{u}(t)\hat{R}(t)^{-1} = \hat{\pi}(t), \tag{4.32}
$$

and α_t to be as in (4.30). To solve Equation (4.29) we are required to make a change of variable

$$
h(t) = [\exp(\int_0^t \delta(s)ds)f(t)]^{\frac{1}{(1-\gamma)}}.
$$

Utilizing this change of variable, we get

$$
\begin{aligned}
f(t) \; &= \; \exp(-\int_0^t \delta(s)ds)\Bigg\{[f(T)^{\frac{1}{(1-\gamma)}}]\{\exp\int_0^T[\frac{\delta(s)}{(1-\gamma)}]ds\} \\
&\times\{\exp\int_t^T[\frac{(\alpha_m-\delta(m))}{(1-\gamma)}]dm\} \\
&+\int_t^T \exp\{-\int_s^t[\frac{(\alpha_m-\delta(m))}{(1-\gamma)}]dm\Bigg\}\,ds\}^{1-\gamma},
\end{aligned} \tag{4.33}
$$

which solves Equation (4.29). Thus, by using (4.32),(4.23) and (4.25), we derive the following expression for $\hat{w}$

$$
\hat{w}(t) = \{\exp\int_0^t[\frac{\delta(s)}{(\gamma-1)}]ds\}f(t)^{\frac{1}{(\gamma-1)}}\hat{R}(t). \tag{4.34}
$$

Moreover, the corresponding Equation (4.18) then becomes

$$\begin{aligned} d\hat{R}(t) &= \hat{R}(t)\{[\rho_t + (\tilde{b}(t) - \rho_t)\hat{\pi}(t) - \{\exp\int_0^t [\frac{\delta(s)}{(\gamma-1)}]ds\}f(t)^{\frac{1}{(\gamma-1)}}]dt \\ &\quad +\sigma_t\hat{\pi}(t)dW(t) + \hat{\pi}(t-)\int_{0<|r|<1} r_1\theta\tilde{N}(dt,dz) \\ &\quad +\hat{\pi}(t-)\int_{|r|\geq 1} r_2\theta N(dt,dz)\}, \end{aligned}$$

with solution being given by

$$\begin{aligned} &\hat{R}(t) \\ &= \hat{R}(0)\exp\{\int_0^t [\rho_s + (\tilde{b}(s) - \rho_s)\hat{\pi}(s) \\ &\quad -\{\exp\int_0^s [\frac{\delta(m)}{(\gamma-1)}]dm\}f(s)^{\frac{1}{(\gamma-1)}} - \frac{1}{2}\hat{\pi}(s)\sigma(s)^2]ds \\ &\quad +\int_0^t \sigma(s)\hat{\pi}(s)dW(s) + \int_0^t (\int_{0<|r|<1} \log(1 + r_1\theta N(ds,dz) \\ &\quad -\int_{0<|r|<1} r_1\theta\nu(x,dz))ds + \int_0^t\int_{|r|\geq 1} r_2\theta N(ds,dz)\}. \end{aligned} \tag{4.35}$$

From this it is reasonable to think that the optimal wealth process will satisfy the terminal condition with equality, as excess wealth is worthless. Although to achieve this we must have as a result of (4.35), f(T) = 0, which in turn gives us, by (4.33)

$$f(t) = \exp(-\int_0^t \delta(s)ds)[\int_t^T \exp\{-\int_s^t [\frac{\alpha_m - \delta(m)}{(1-\gamma)}]dm\}]^{1-\gamma}. \tag{4.36}$$

Hence we have

$$f(s) \sim (T-s)^{1-\gamma}, \qquad as \quad s \to T-.$$

Therefore

$$\int^T f(s)^{\frac{1}{(\gamma-1)}}ds \sim \int^T (T-s)^{-1}ds = \infty,$$

which by (4.35) gives that $\hat{R}(T) = 0$, as required. Thus all the conditions of the maximum principle are satisfied and so is the terminal condition (4.20).

4.3.2 The Wealth-consumption portfolio optimization problem.

For this we consider the construction when $d = 1$ and we refer to Theorem 4.2.2, thus we have the resulting relation

$$\begin{aligned}\int_U 1_A(c(x,y))\lambda(dy) &= \int_{\mathbb{R}\setminus\{0\}} 1_A(z)\nu(x,dz) \\ &= \int_{\mathbb{R}\setminus\{0\}} 1_A(z)\frac{dz}{|z|^{1+\alpha(x)}} \\ &= \int_{(0,\infty)} 1_A(\rho)\frac{d\rho}{\rho^{1+\alpha(x)}}\end{aligned}$$

This equivalence can be made since we know that $z = \rho$ and that when $d = 1$ there is no angular dependence so the finite Borel measure $S(d\theta)$ is a constant. Once again for this problem we will assume all the conditions outlined in section 1.3.2 are adhered to.

So the coefficient of the jump term is

$$\hat{c}(r) = r, \quad r \in (0,\infty).$$

Let us start with a Lévy-type process Z_t defined by the Lévy-Ito decomposition

$$Z_t = \mu t + \int_0^t \phi(s)dW_s + \int_0^t\int_{0<|r|<1} r_1\tilde{N}(ds,dz) + \int_0^t\int_{|r|\geq 1} r_2 N(ds,dz)$$

where μ is a constant, $\phi : [0,T] \to \mathbb{R}$, $r_1, r_2 \in (0,\infty)$. Where we have a compensated Poisson random measure

$$\tilde{N}(dt,dz) := N(dt,dz) - dt\nu(x,dz).$$

Throughout this section, we will assume that

$$\int_{|r|\geq 1} (e^{r_2} - 1)\nu(x, dz) < \infty. \tag{4.37}$$

Starting from this, the jump type SDE we are concerned is formulated in the following manner

$$dS_t = b(t, S_t)dt + \sigma(t, S_t)dZ_t,$$

where $b : [0, T] \times \mathbb{R} \to \mathbb{R}$,$\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ are measurable.

If we allow $b(t, S_t) = b(t)S_t$ and $\sigma(t, S_t) = \sigma(t)S_t$, i.e., the coefficients b, σ are linear on S_t,then we get a geometric Lévy-type process $S_t = S_0 e^{Z_t}$ with initial data $S_0 > 0$.

By the Itô formula (cf. e.g. Theorem II.5.1 in [25]), we get the following equation for S_t

$$\begin{aligned} dS_t &= b(t)S_t dt + \frac{1}{2}\sigma(t)^2 S_t dt + \sigma(t)S_t dW_t \\ &+S_t \int_{0<|r|<1} (e^{r_1} - 1 - r_1)\nu(x, dz)dt \\ &+S_{t-} \int_{0<|r|<1} (e^{r_1} - 1)\tilde{N}(dt, dz) + S_{t-} \int_{|r|\geq 1} (e^{r_2} - 1)N(dt, dz). \end{aligned} \tag{4.38}$$

Let D_β be defined

$$D_\beta := \{(x, y); y > 0, y + \beta x > 0\},$$

where $y = 0$ and $y + \beta x = 0$ are the lower boundaries of D_β, $\bar{D}_\beta$ is the closure of D_β and let $\beta > 0$ be the damping rate of the average past consumption.

From chapter 2 we know the wealth process X_t and the average past consumption process $Y_t > 0$ are constructed based on the processes Z_t and the asset price S_t, both being adapted. These processes X_t, Y_t are dependent on 3 parameter processes which represent the control, namely (π_t, G_t, L_t).

Let $\pi_t \in [0,1]$ be the fraction of the wealth that is invested in the asset S_t, let G_t be the cumulative consumption up to time t, note this is only enforced when the wealth is non-negative. The process L_t is a control used to adjust wealth should it become negative, for example when additional income is recieved. These controls are assumed to be admissible provided they satisfy the conditons outlined in section 2.3.

Now the wealth process X_t for our concrete situation can be defined from considering a self-financing investment policy according to the portfolio π_t:

$$\frac{dX_t}{X_{t-}} = (1-\pi_t)\frac{dB_t}{B_t} + \pi_t \frac{dS_t}{S_{t-}}$$

where B_t denotes the riskless bond given by $dB_t = qB_t dt$.

$$\begin{aligned}
X_t &= x - G_t + \int_0^t \sigma(s)\pi_s X_s dW_s + L_t \\
&+ \int_0^t (q + ([b(s) + \frac{1}{2}\sigma(s)^2 + \int_{0<|r|<1} (e^{r_1} - 1 - r_1)\nu(x,dz)] - q)\pi_s)X_s ds \\
&+ \int_0^t \pi_{s-}X_{s-} \int_{0<|r|<1} (e^{r_1} - 1)\tilde{N}(ds,dz) \\
&+ \int_0^t \pi_{s-}X_{s-} \int_{|r|\geq 1} (e^{r_2} - 1)N(ds,dz).
\end{aligned} \tag{4.39}$$

The process Y_t can be constructed from $dY_t = -\beta Y_t dt + \beta dG_t$, namely

$$Y_t = ye^{-\beta t} + \beta \int_0^t e^{-\beta(t-s)} dG_s. \tag{4.40}$$

For such a model we fix the parameters (π_t, g_t, L_t) such that we have a Markovian control. Also if we allow the value to be determined by the value of the pair (X_t, Y_t); then the value fundtion can be defined by

$$v(x,y) = \sup_{(\pi_\cdot, g_\cdot, L_\cdot)\in\mathcal{A}} \mathbb{E}^{(X^{(\pi_\cdot,g_\cdot,L_\cdot)},Y^{(\pi_\cdot,g_\cdot,L_\cdot)})}[\int_0^\infty e^{-\alpha s}u(Y_s)ds], \tag{4.41}$$

where $X_t^{(\pi.,g.,L.)}, Y_t^{(\pi.,g.,L.)}$ are the processes X_t, Y_t given $(\pi., g., L.)$, and $\alpha > 0$ is the dumping rate of the utility.

The aim is to characterize the value function v as a viscosity solution to the Hamilton-Jacobi-Bellman(HJB) equation

$$\begin{aligned} &\max\{Nv, \sup_{\pi,g\in\mathcal{A}}\{Av\}, Mv\} = 0 \quad in\ D_\beta, \\ &v = 0 \quad outside\ of\ D_\beta. \end{aligned} \tag{4.42}$$

where

$$Nv = v_x.1_{\{x\leq 0\}},$$

and

$$Mv = (\beta v_y - v_x).1_{\{x\geq 0\}}.$$

Let v_x, v_{xx}, v_y denote the partial derivatives of order 1 or 2 with respect to x and y and the operators M and N correspond to the continuous parts of the controls G_t, L_t in (4.39) and (4.40).

Let the generator A that is associated with the pair (X_t, Y_t) be defined

$$\begin{aligned} Av(x,y) &= -\alpha v - \beta y v_y + \sigma(t)\pi x v_{xx} \\ &+\{(q + \pi([b(t) + \frac{1}{2}\sigma(t)^2 + \int_{0<|r|<1}(e^{r_1} - 1 - r_1)\nu(x,dz)] - q))x v_x \\ &+ \int_{0<|r|<1}(v(x + \pi x(e^{r_1} - 1), y) - v(x,y) - \pi x v_x(e^{r_1} - 1))\nu(x,dz) \\ &+ \int_{|r|\geq 1}(v(x + \pi x(e^{r_2} - 1), y) - v(x,y))\nu(x,dz)\} \\ &+u(y) - y(v_x - \beta v_y), \qquad \pi \in [0,1]. \end{aligned} \tag{4.43}$$

Where the function space associated to this generator A, namely $Q(\bar{D}_\beta)$ consists of bounded and integrable functions which are twice differentiable for the variables x, y and is convergent due to the property of Lévy measures.

By the defintions of sub, super and strict viscosity solutions of the integro-differential HJB equation (4.42) outlined in section 3.1, we can state the following existence and uniqueness theorems.

Theorem 4.3.1 *The well defined and bounded value function v(x,y) is a constrained viscosity solution of (4.42).*

Theorem 4.3.2 *For each $\gamma > 0, \mu \geq 0$ choose $\alpha > 0$ so that $\alpha > k(\gamma, \mu)$, where we define a finite constant $k(\gamma, \mu)$ by*

$$
\begin{aligned}
& k(\gamma, \mu) \\
= \; & \max[\gamma(q + \pi([b(t) + \frac{1}{2}\sigma(t)^2 + \int_{0<|r|<1} (e^{r_1} - 1 - r_1)\lambda(dz)] - q)) \\
& +\sigma(t)\pi\mu + \int_{0<|r|<1} [(1 + \pi(e^{r_1} - 1))^{\gamma} - 1 - \gamma\pi(e^{r_1} - 1)) \\
& + \int_{|r|\geq 1} (1 + \pi(e^{r_2} - 1))^{\gamma} - 1]\nu(x, dz)].
\end{aligned}
$$

Now assuming that $v_0 \in Q(\bar{D}_\beta)$ is a subsolution of (4.42) in $\bar{D}_\beta$ and $\bar{v} \in Q(\bar{D}_\beta)$ is a supersolution of (4.42) in D_β. Then

$$v_0 \leq \bar{v} \quad \text{on } \bar{D}_\beta.$$

Consequently,the HJB equation admits at most one constrained viscosity solution in $Q(\bar{D}_\beta)$.

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